浙江理工大学学报，2022，47(4)：579-587
Journal of Zhejiang Sci-Tech University
DOI：10.3969/j.issn.1673-3851(n).2022.04.015

# 广义 Grötzsch 函数的一些函数不等式

林　杰,张孝惠
（浙江理工大学理学院,杭州 310018）

**摘　要**：通过研究涉及$(p,q)$-完全椭圆积分、$(p,q)$-Grötzsch 函数 $\mu_{p,q}(r)$与一些初等函数的特殊函数的单调性和凹凸性,给出了$(p,q)$-Grötzsch 函数的一些精确的初等逼近和函数不等式,从而推广和改进了经典 Grötzsch 函数的一些已知结果,如几何调和凹性和 $r=0$ 处的对数奇性等。这些逼近和不等式可应用于广义偏差函数的估计和 Ramanujan 模方程的研究。

**关键词**：完全$(p,q)$-椭圆积分;$(p,q)$-Grötzsch 函数;单调性;凹凸性;不等式
**中图分类号**：O174.6　　**文献标志码**：A　　**文章编号**：1673-3851（2022）07-0579-09

# Some functional inequalities for the generalized Grötzsch function

*LIN Jie*，*ZHANG Xiaohui*
（School of Science，Zhejiang Sci-Tech University，Hangzhou 310018，China）

**Abstract**：By investigating the monotonicity and concavity and convexity of some special functions involving the complete $(p,q)$-elliptic integral，the $(p,q)$-Grötzsch function $\mu_{p,q}(r)$，and some elementary functions，the authors presented some sharp elementary approximations and functional inequalities for the $(p,q)$-Grötzsch function，thus promoting and improving several well-known results for the classical Grötzsch function，such as the property of geometric-harmonic concavity and the logarithmic singularity at $r=0$. These approximations and inequalities can be applied to the estimation of generalized distortion functions and the study of Ramanujan′s modular equations.

**Key words**：complete $(p,q)$-elliptic integral；$(p,q)$-Grötzsch function；monotonicity；concavity and convexity；inequality

## 0　引　言

## 0　Introduction

For given real numbers $a,b$ and $c$ with $c\neq 0,-1,-2\cdots$，

$$ {}_2F_1(a,b,c,x)\equiv\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!},\quad |x|<1 $$

is the classical Gaussian hypergeometric function，where $(a,0)=1$ for $a\neq 0$，and $(a,n)\equiv a(a+1)(a+2)\cdots(a+n-1)$ for positive integer $n$.

For $p,q\in(1,\infty)$ and $r\in(0,1)$，the generalized $(p,q)$-elliptic integrals can be represented by the Gaussian hypergeometric function[1]：

收稿日期：2021－11－12　　网络出版日期：2022－01－05
基金项目：国家自然科学基金项目(11771400)
作者简介：林　杰(1996—　)，女，河南信阳人，硕士研究生，主要从事拟共形理论、特殊函数、Ramanujan 模方程方面的研究。
通信作者：张孝惠，E-mail：xiaohui.zhang@zstu.edu.cn

$$\begin{cases}\kappa_{p,q}=\kappa_{p,q}(r)=\dfrac{\pi_{p,q}}{2}\,{}_2F_1\left(1-\dfrac{1}{p},\dfrac{1}{q},1-\dfrac{1}{p}+\dfrac{1}{q};r^q\right),\\ \kappa'_{p,q}=\kappa'_{p,q}(r)=\kappa_{p,q}(r'),\\ \kappa_{p,q}(0)=\dfrac{\pi_{p,q}}{2},\kappa_{p,q}(1)=\infty\end{cases}$$

and

$$\begin{cases}\varepsilon_{p,q}=\varepsilon_{p,q}(r)=\dfrac{\pi_{p,q}}{2}\,{}_2F_1\left(-\dfrac{1}{p},\dfrac{1}{q},1-\dfrac{1}{p}+\dfrac{1}{q};r^q\right),\\ \varepsilon'_{p,q}=\varepsilon'_{p,q}(r)=\varepsilon_{p,q}(r'),\\ \varepsilon_{p,q}(0)=\dfrac{\pi_{p,q}}{2},\varepsilon_{p,q}(1)=1.\end{cases}$$

Here and hereafter, we always let $r'=(1-r^q)^{1/q}$. It is easy to see that $\kappa_{p,q}$ is strictly increasing and $\varepsilon_{p,q}$ is strictly decreasing on $(0,1)$, respectively. The generalized $(p,q)$-elliptic integrals satisfy the following beautiful Legendre relation in Corollary 1. 2 in [2]:

$$\kappa_{p,q}(r)\varepsilon'_{p,q}(r)+\kappa'_{p,q}(r)\varepsilon_{p,q}(r)-\kappa_{p,q}(r)\kappa'_{p,q}(r)=\frac{\pi_{p,q}}{2} \tag{1}$$

For $p=q=2$, the generalized $(p,q)$-elliptic integrals reduce to the classical complete elliptic integrals, respectively. It is well known that the complete elliptic integrals have an important role in several branches of mathematics, such as special function theory and number theory, as well as in physics and engineering. Numerous properties have been obtained for $\kappa$ and $\varepsilon$ (for instance, cf. [3−7]). However, only a few basic properties of the generalized $(p,q)$-elliptic integrals $\kappa_{p,q}$ and $\varepsilon_{p,q}$ have been revealed[1-2, 8]. It is natural to ask whether the known properties of $\kappa$ and $\varepsilon$ can be extended to the generalized functions $\kappa_{p,q}$ and $\varepsilon_{p,q}$.

We define three related functions $\mu_{p,q}, m_{p,q}, M_{p,q}$ as follows: for $p$, $q\in(1,\infty)$ and $r\in(0,1)$,

$$\mu_{p,q}(r)=\frac{\pi_{p,q}}{2}\frac{\kappa'_{p,q}(r)}{\kappa_{p,q}(r)},m_{p,q}(r)=\frac{2}{\pi_{p,q}}r'^q\kappa_{p,q}(r)\kappa'_{p,q}(r),M_{p,q}(r)=m_{p,q}(r)+\log r \tag{2}$$

For $p=q=2$, these functions reduce to well-known special cases. The function $\mu(r)=\mu_{2,2}(r)$ is the modulus of the Grötzsch ring domain in the plane, which has numerous applications in the conformal invariants and the theory of quasiconformal mappings[3,9]. The classical modular equations can also be represented by the Grötzsch ring function $\mu(r)$[6,10]. In the monograph [3], the authors presented many sharp elementary approximations and elegant functional inequalities for the Grötzsch function by studying monotonicity and convexity properties of functions involving the Grötzsch function.

The purpose of this paper is to extend some well-known results for the function $\mu(r)$ to the generalized function $\mu_{p,q}(r)$ by investigating some monotonicity and convexity properties of certain combinations defined in terms of the generalized $(p,q)$-elliptic integrals, the generalized Grötzsch ring function $\mu_{p,q}(r)$, and some elementary functions. Our main results are stated in the next section.

## 1 主要结果

## 1 Main results

Throughout this paper, we always set $p,q>1$, $a=1-1/p$, $b=1/q$, $c=a+b$, $d=(p-2+\sqrt{5p^2-8p+4})/(2(p-1))$, $D=(2p-3+\sqrt{8p^2-16p+9})/(2(p-1))$. Let $\gamma=0.577215\cdots$ be the Euler constant and $\psi$ be the classical psi function, and let

$$R(a,b)=-2\gamma-\psi(a)-\psi(b), R(a)=R(a,1-a), R(1/2)=\log 16.$$

In this section, we state our main results which show some monotonicity and convexity properties of some functions involving the generalized Grötzsch function, and obtain some sharp functional inequalities for the generalized Grötzsch function.

Our first theorem extends the results of Theorem 1.2 in [11].

**Theorem 1**　a) When $q\geqslant(3p-4)/(p-1)$, the function $h_1(r)=(r'^{q/2}\mu_{p,q}(r))/\log(1/r)$ is strictly increasing from $(0,1)$ onto $(1,\infty)$.

b) When $q\geqslant d$, the function $h_2(r)=\mu_{p,q}(r)/(bR(a,b)+\log(1/r))$ is strictly decreasing and concave on $(0,1)$ with range $(0,1)$.

c) When $q\geqslant(3p-4)/(p-1)$, the function $h_3(r)=h_2(r)/r'^{q/4}$ is strictly increasing from $(0,1)$ onto $(1,\infty)$. Moreover,

$$(bR(a,b)-\log r)r'^{q/4}<\mu_{p,q}(r)<bR(a,b)-\log r.$$

**Theorem 2**　a) If and only if $p>3/2$ and $q\leqslant(3p-4)/(p-1)$, $g_1(r)=\mu_{p,q}(r)+\log(r/\sqrt{r'})$ is strictly increasing and convex on $(0,1)$ with range $(bR(a,b),\infty)$.

b) For $0<r<1/2^{1/q}$, $g_2(r)=(\mu_{p,q}(r)-\pi_{p,q}/2)/\log(r/r')$ is decreasing from $(0,1/2^{1/q})$ onto $(\pi_{p,q}/A,1)$, where $A=4\kappa_{p,q}(1/2^{1/q})^2/\pi_{p,q}$. In particular, for all $0<r<1/2^{1/q}$,

$$\frac{\pi_{p,q}}{2}+\frac{\pi_{p,q}}{A}\log\left(\frac{r'}{r}\right)<\mu_{p,q}(r)<\frac{\pi_{p,q}}{2}+\log\left(\frac{r'}{r}\right),$$

and for all $1/2^{1/q}<r<1$,

$$\frac{\pi_{p,q}^2}{4(\pi_{p,q}/2+\log(r/r'))}<\mu_{p,q}(r)<\frac{\pi_{p,q}^2}{4(\pi_{p,q}/2+(\pi_{p,q}/A)\log(r/r'))}.$$

c) When $q\geqslant(3p-4)/(p-1)$, the function $g_3(r)=\mu_{p,q}(r)\log(1/(1-r^{q/2}))$ is increasing from $(0,1)$ onto $(0,\ q\pi_{p,q}^2/4)$. In particular, for all $0<r<1$,

$$\mu_{p,q}(r)\leqslant\frac{q\pi_{p,q}^2}{4\log(1/(1-r^{q/2}))}.$$

The following Theorem 3 and Theorem 4 are extensions of Theorem 1.3 in [11] and Theorem 1.28 in [12], respectively.

**Theorem 3**　a) The function $G_1(r)=\mu_{p,q}(1/r)$ is strictly increasing and concave from $(1,\infty)$ onto $(0,\infty)$. In particular, for $x,y,\lambda\in(0,1)$,

$$\lambda\mu_{p,q}(x)+(1-\lambda)\mu_{p,q}(y)\leqslant\mu_{p,q}\left(\frac{xy}{\lambda y+(1-\lambda)x}\right)$$

with equality if and only if $x=y$.

b) For each $t\in(0,1)$, the function $G_2(r)=\mu_{p,q}(rt/(1+t'))-\mu_{p,q}(r)$ is strictly increasing from $(0,1)$ onto $(\mathrm{arth}t',\ \mu_{p,q}(t/(1+t')))$. Moreover, for $r,t\in(0,1)$,

$$\mu_{p,q}(r)+\mathrm{arth}t'<\mu_{p,q}(rt/(1+t'))<\mu_{p,q}(r)+\mu_{p,q}(t/(1+t')).$$

c) The function $G_3(r)=\mu_{p,q}(r)/\mu_{p,q}(\sqrt{r})$ is strictly decreasing from $(0,1)$ onto $(1,2)$. In particular, for all $p,q>1$ and $r\in(0,1)$,

$$\mu_{p,q}(r)<\mu_{p,q}(r^2)<2\mu_{p,q}(r).$$

**Theorem 4**　a) The function $H_1(r)=((r'^q\log r')/(r^q\log r))\mu_{p,q}(r)$ is strictly increasing from $(0,1)$ onto $(1/q,q\pi_{p,q}^2/4)$. In particular, for all $r\in(0,1)$,

$$\frac{1}{q}\frac{r^q\log r}{r'^q\log r'}<\mu_{p,q}(r)<q\left(\frac{\pi_{p,q}}{2}\right)^2\frac{r^q\log r}{r'^q\log r'}.$$

b) When $q\geqslant2$, $H_2(r)=((r'^{q/2}\mathrm{arth}r^{q/2})/(r^{q/2}\mathrm{arth}r'^{q/2}))\mu_{p,q}(r)$ is strictly increasing from $(0,1)$ onto

$(2/q,\ q\pi_{p,q}^2/8)$. In particular, for all $r\in(0,1)$,

$$\frac{2}{q}\frac{r^{q/2}\mathrm{arth}r'^{q/2}}{r'^{q/2}\mathrm{arth}r^{q/2}}<\mu_{p,q}(r)<\frac{q}{2}\left(\frac{\pi_{p,q}}{2}\right)^2\frac{r^{q/2}\mathrm{arth}r'^{q/2}}{r'^{q/2}\mathrm{arth}r^{q/2}}.$$

## 2 数学基础

## 2 Preliminary and derivatives

The following two lemmas are very useful in proving monotonicity of the ratio of two functions or two series, and Lemma 1 and Lemma 2 are from Theorem 1.25 in [3] and [13], respectively.

**Lemma 1** Let $-\infty<a<b<+\infty$, and $f,g:[a,b]\to\mathbf{R}$ be continuous functions defined in $[a,b]$ and differentiable in $(a,b)$, and suppose that $g'(x)\neq 0$ on $(a,b)$. If $f'(x)/g'(x)$ is increasing (decreasing, respectively) on $(a,b)$, then so are the functions

$$\frac{f(x)-f(a)}{g(x)-g(a)}\quad\text{and}\quad\frac{f(x)-f(b)}{g(x)-g(b)}\tag{3}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity of the functions in (3) is also strict.

**Lemma 2** Let $r_n$ and $s_n$ for $n\in\mathbb{N}$ be real numbers, and suppose that the power series

$$f(x)=\sum_{n=0}^{\infty}a_nx^n\quad\text{and}\quad g(x)=\sum_{n=0}^{\infty}b_nx^n$$

are both convergent for $x\in(-1,1)$. If $b_n>0$ for all $n\in\mathbb{N}$ and if $a_n/b_n$ is strictly increasing (or decreasing, respectively), then the function $f(x)/g(x)$ is strictly increasing (or decreasing, respectively) on $(-1,1)$.

The functions $\kappa_{p,q}$ and $\varepsilon_{p,q}$ satisfy a system of differential equations[1]:

$$\frac{\mathrm{d}\kappa_{p,q}}{\mathrm{d}r}=\frac{\varepsilon_{p,q}-r'^q\kappa_{p,q}}{rr'^q},\frac{\mathrm{d}\varepsilon_{p,q}}{\mathrm{d}r}=\frac{q(\varepsilon_{p,q}-\kappa_{p,q})}{pr}\tag{4}$$

From (4), we have the following derivative formulas:

$$\frac{\mathrm{d}(\varepsilon_{p,q}-r'^q\kappa_{p,q})}{\mathrm{d}r}=\frac{(p-q)(\kappa_{p,q}-\varepsilon_{p,q})+p(q-1)r^q\kappa_{p,q}}{pr}\tag{5}$$

$$\frac{\mathrm{d}(\kappa_{p,q}-\varepsilon_{p,q})}{\mathrm{d}r}=\frac{(p-qr'^q)\varepsilon_{p,q}+(q-p)r'^q\kappa V p,q}{prr'^q}\tag{6}$$

By the definitions (2), and the derivative formulas (4), we obtain the derivative formulas for the functions $\mu_{p,q}$ and $m_{p,q}$ as the following lemma shows.

**Lemma 3** Let $p,q>1$, for $0<r<1$,

$$\frac{\mathrm{d}\mu_{p,q}(r)}{\mathrm{d}r}=\frac{\pi_{p,q}^2}{-4rr'^q\kappa_{p,q}^2}\tag{7}$$

$$\frac{\mathrm{d}m_{p,q}(r)}{\mathrm{d}r}=-\frac{1}{r}-\frac{4}{\pi_{p,q}}r^{q-1}\kappa_{p,q}\kappa'_{p,q}\left(\frac{q}{2}-\frac{\varepsilon_{p,q}-r'^q\kappa_{p,q}}{r^q\kappa_{p,q}}\right)\tag{8}$$

**Lemma 4** For $r,t\in(0,1)$, we have the following conclusions:

a) The function $f_1(r)=r'^c\kappa_{p,q}(r)$ is strictly decreasing from $(0,1)$ onto $(0,\pi_{p,q}/2)$ if and only if $c\geqslant(q(p-1))/(q(p-1)+p)$.

b) The function $f_2(r)=(\varepsilon_{p,q}(r)-r'^q\kappa_{p,q}(r))/(r^q\kappa_{p,q}(r))$ is strictly decreasing from $(0,1)$ onto $(0,(q(p-1))/(q(p-1)+p))$.

c) The function $f_3(r)=\kappa_{p,q}(r)+\log r'$ is strictly decreasing and concave from $(0,1)$ onto $(R(a,b)/q,\pi_{p,q}/2)$.

d) The function $f_4(r)=(\varepsilon_{p,q}(r)-r'^q\kappa_{p,q}(r))/(\kappa_{p,q}(r)-\varepsilon_{p,q}(r))$ is strictly decreasing from $(0,1)$

onto $(0,q(p-1)/p)$.

e) When $q\geqslant 2$, $f_5(r)=\operatorname{arth}r^{q/2}/(r^{q/2}\kappa_{p,q}(r))$ is strictly increasing from $(0,1)$ onto $(2/\pi_{p,q},q/2)$.

f) The function $f_6(r)=r'^{q/2}\exp(qr^q\kappa_{p,q}(r)/(2(\varepsilon_{p,q}-r'^q\kappa_{p,q}(r))))$ is strictly decreasing from $(0,1)$ onto $(\exp(R(a,b)/2),\exp((q(p-1))/(q(p-1)+p)))$.

g) Let $c\geqslant 1$, $f_7(r)=\kappa_{p,q}(r)/\log(c/r'^{q/2})$ is strictly decreasing if $c\leqslant\exp(R(a,b)/2)$ and strictly increasing if $c\geqslant\exp((q(p-1)+p)/(2(p-1)))$. In particular, $\kappa_{p,q}(r)/\log(c/r'^{q/2})$ lies between $q/2$ and $\pi_{p,q}/(2\log c)$.

**Proof** The proof of (a)—(d) can be referred to [14].

e) Let $f_{51}(r)=\operatorname{arth}r^{q/2}$, $f_{52}(r)=r^{q/2}\kappa_{p,q}(r)$, then $f_{51}(0)=f_{52}(0)=0$ and

$$\frac{f'_{51}(r)}{f'_{52}(r)}=\frac{1}{2\varepsilon_{p,q}/q+(1-2/q)r'^q\kappa_{p,q}}.$$

When $q\geqslant 2$, the monotonicity of $f_5(r)$ follows from Lemma 4(a). Clearly, $f_5(1^-)=q/2$, $f_5(0^+)=2/\pi_{p,q}$.

f) Let

$$G(r)=\log(f_6(r))=\frac{q\log r'}{2}+\frac{qr^q\kappa_{p,q}}{2(\varepsilon_{p,q}-r'^q\kappa_{p,q})}.$$

By differentiation, it follows from (4) and (5) that

$$G'(r)=\frac{qr^{q-1}\kappa_{p,q}(p(\varepsilon_{p,q}-r'^q\kappa_{p,q})-q(p-1)(\kappa_{p,q}-\varepsilon_{p,q}))}{2p\,(\varepsilon_{p,q}-r'^q\kappa_{p,q})^2}.$$

Let

$$F(r)=\frac{pq(\varepsilon_{p,q}-r'^q\kappa_{p,q})}{(p-1)q^2(\kappa_{p,q}-\varepsilon_{p,q})}=\frac{p}{q(p-1)}\,\frac{\varepsilon_{p,q}-r'^q\kappa_{p,q}}{\kappa_{p,q}-\varepsilon_{p,q}}.$$

Then $F(r)$ is strictly decreasing from $(0,1)$ onto $(0,1)$ by Lemma 4(d) and hence $p(\varepsilon_{p,q}-r'^q\kappa_{p,q})<q(p-1)(\kappa_{p,q}-\varepsilon_{p,q})$. Therefore, $G'(r)<0$, and the monotonicity of $G(r)$ follows. Consequently, we see the monotonicity of $f_6(r)$. The limiting values follow from Lemma 4(c)

$$\begin{aligned}\lim_{r\to 1^-}\frac{2}{q}\log(f_6(r))&=\lim_{r\to 1^-}\left(\frac{r^q(\kappa_{p,q}+\log r')}{\varepsilon_{p,q}-r'^q\kappa_{p,q}}+\left(1-\frac{r^q}{\varepsilon_{p,q}-r'^q\kappa_{p,q}}\right)\log r'\right)\\&=\frac{R(a,b)}{q}+\lim_{r\to 1^-}(\varepsilon_{p,q}-r'^q\kappa_{p,q}-r^q)\log r'\\&=\frac{R(a,b)}{q}+\lim_{r\to 1^-}\left(\frac{\varepsilon_{p,q}-r^q}{}-r'^{q-1}\kappa_{p,q}\right)r'\log r'.\end{aligned}$$

Next, by Theorem 1.19(6) in [3] and (1),

$$\lim_{r\to 1^-}r'^{q-1}\kappa_{p,q}(r)=\lim_{r\to 1^-}\frac{\pi_{p,q}}{2}\frac{-\Gamma(a+b)}{\Gamma(a)\Gamma(b)}r'^{q-1}\log(1-r^q)=0,$$

then we have $\lim_{r\to 1^-}2/q\log(f_6(r))=R(a,b)/q$. Therefore

$$f_6(1^-)=\exp\left(\frac{R(a,b)}{2}\right),\ f_6(0^+)=\exp\left(\frac{q(p-1)+p}{2(p-1)}\right).$$

g) By differentiation,

$$f'_7(r)=\frac{\dfrac{\varepsilon_{p,q}-r'^q\kappa_{p,q}}{rr'^q}\log\left(\dfrac{c}{r'^{q/2}}\right)-\dfrac{qr^{q-1}\kappa_{p,q}}{2r'^q}}{(\log(c/r'^{q/2}))^2},$$

$f'_7(r)$ is negative or positive according as

$$c \leqslant \inf\left\{r'^{q/2}\exp\left(\frac{qr^q\kappa_{p,q}}{2(\varepsilon_{p,q}-r'^q\kappa_{p,q})}\right)\right\}=\exp\left(\frac{R(a,b)}{2}\right),$$

$$c \geqslant \sup\left\{r'^{q/2}\exp\left(\frac{qr^q\kappa_{p,q}}{2(\varepsilon_{p,q}-r'^q\kappa_{p,q})}\right)\right\}=\exp\left(\frac{q(p-1)+p}{2(p-1)}\right).$$

By simple calculation, the limiting values follow from Lemma 4(b)

$$f_7(0^+)=\frac{\pi_{p,q}}{2\log c}, f_7(1^-)=\lim_{r\to 1^-}\frac{\varepsilon_{p,q}-r'^q\kappa_{p,q}}{rr'^q}\frac{2r'^q}{qr^{q-1}}=\frac{2}{q}.$$

The following lemma shows monotonicity properties of the functions $\mu_{p,q}(r)$ and $m_{p,q}(r)$, see [14-15].

**Lemma 5** For $r\in(0,1)$, we have the following conclusions:

a) The function $f_8(r)=\mu_{p,q}(r)+\log r$ is strictly decreasing from $(0,1)$ onto $(0,bR(a,b))$.

b) The function $f_9(r)=m_{p,q}(r)/\log(1/r)$ is strictly increasing from $(0,1)$ onto $(1,\infty)$.

c) The function $f_{10}(r)=m_{p,q}(r)+\log r$ is strictly decreasing from $(0,1)$ onto $(0,bR(a,b))$. Moreover, if $q\geqslant d$, then $f_{10}(r)$ is concave on $(0,1)$.

d) The function $f_{11}(r)=m_{p,q}(rt)-m_{p,q}(r)$ is strictly increasing from $(0,1)$ onto $(-\log t, m_{p,q}(t))$.

The monotonicity properties of the functions $f_{12}, f_{13}$ and $f_{14}$ are from Theorem 2.3 in [11], Lemma 2.16(2) in [12] and Theorem 1.4(1) in [16], respectively.

**Lemma 6** Let $a_1, b_1\in(0,\infty)$ with $c_1=a_1+b_1$, we have the following conclusions:

a) The function $f_{12}(x)=F(a_1,b_1,c_1;x)/(R(a_1,b_1)-\log(1-x))$ is strictly decreasing from $[0,1)$ onto $(1/B(a_1,b_1),1/R(a_1,b_1)]$.

b) For $2a_1b_1\leqslant c_1$, the function $f_{13}(x)=(1-(1-x)F(a_1,b_1,c_1;x)^2)/x$ is strictly increasing from $(0,1)$ onto $(1-(2a_1b_1/c_1),1)$.

c) If $a_1,b_1\in(0,1)$, then $f_{14}(x)=xF(a_1,b_1,c_1;x)/\log(1/(1-x))$ is decreasing from $(0,1)$ onto $(1/B(a_1,b_1),(1)$.

## 3 主要结果的证明

## 3 Proofs of main results

### 3.1 定理1的证明

### 3.1 Proof of Theorem 1

a) First, we observe that $h_1(r)$ can be rewritten as

$$h_1(r)=\frac{m_{p,q}(r)}{\log(1/r)}\frac{1}{r'^{q/2}F(a,b,c;r^q)^2}$$

which is strictly increasing by Lemma 4(a) and Lemma 5(b), since

$$\frac{q}{4}\geqslant\frac{q(p-1)}{q(p-1)+p}.$$

Therefore, the monotonicity of $h_1(r)$ follows and the limiting values are clear.

b) We have the derivative

$$-h_2'(r)=\frac{\pi_{p,q}^2}{(2(bR(a,b)+\log(1/r))r'^{q/2}\kappa_{p,q})^2}\frac{bR(a,b)-(m_{p,q}(r)+\log r)}{r}$$

which is a product of two positive and strictly increasing functions on $(0,1)$ by Lemma 4(a) and Lemma 5(c). Hence the monotonicity of $h_2(r)$ follows. Clearly, $h_2(1^-)=0$, while the limit $h_2(0^+)=1$ follows from Theorem 1(a).

c) $h_3(r)$ can be written as

$$h_3(r)=\frac{\mu_{p,q}(r)}{b(R(a,b)-\log r^q)r'^{q/4}}=\frac{q\pi_{p,q}}{2r'^{q/4}\kappa_{p,q}}\frac{\kappa'_{p,q}}{R(a,b)-\log r^q}.$$

The monotonicity of $h_3(r)$ follows from Lemma 4(a) and Lemma 6(a). Moreover, the limiting values are obvious.

### 3.2　定理 2 的证明

### 3.2　Proof of Theorem 2

a) By differentiation

$$g_1'(r)=\frac{(2-r^q)F(a,b,c;r^q)^2-2}{2rr'^qF(a,b,c;r^q)^2}.$$

Let $x=r^q$ then $g_1'(r)=f(x)/(2x^{1/q}r'^qF(a,b,c;r^q)^2)=F(x)/(2r'^qF(a,b,c;r^q)^2)$, where $f(x)=(2-x)F(a,b,c,x)^2-2$, $F(x)=f(x)/x^{1/q}$. Clearly, $f(0^+)=0$, $f(1^-)=\infty$. It suffices to have $f'(x)>0$. We have $f'(x)=F(a,b,c,x)g(x)$ by (1.16) in [3], where

$$g(x)=\frac{ab}{c}2(2-x)F(a+1,b+1,c+1;x)-F(a,b,c;x).$$

Using series expansion of $F(a,b,c,x)$, we get

$$\begin{aligned}g(x)&=(4-2x)\sum_{n=0}^{\infty}\frac{(a,n+1)(b,n+1)}{(c,n+1)n!}x^n-\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)n!}x^n\\&=4\sum_{n=0}^{\infty}\frac{(a+n)(b+n)(a,n)(b,n)}{(c,n+1)n!}x^n-2\sum_{n=0}^{\infty}\frac{n(a,n)(b,n)}{(c,n)n!}x^n-\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)n!}x^n\\&=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n+1)n!}(2n^2+(2(a+b)-1)n+4ab-a-b)x^n.\end{aligned}$$

Since $2n^2+(2(a+b)-1)n+4ab-a-b>0$ for $n\geqslant 1$, we see that $g(x)$ is strictly increasing on $(0,1)$. Clearly, $g(0^+)=(4ab-a-b)/c$. By Theorem 1.19(4) in [3] and Lemma 4(a), we find

$$g(1^-)=\sum_{x\to 1^-}\frac{1}{1-x}\left\{\frac{ab}{c}(4-2x)F(a,b,c+1;x)-(1-x)F(a,b,c;x)\right\}=\infty.$$

As the monotonicity of $g(x)$, $f(x)>0$ if and only if $g(0^+)\geqslant 0$ which is equivalent to $p>3/2$ and $q\leqslant(3p-4)/(p-1)$. Next, we prove the convexity of $g_1(r)$, let $h(x)=r=x^{1/q}$, then $F(x)=f(x)/h(x)$, $f(0)=h(0)=0$ and $f'(x)/h'(x)=qx^{(q-1)/q}F(a,b,c,x)g(x)$. Hence, we get the monotonicity and convexity of $g_1(r)$ by Lemma 5(a). Clearly, $g_1(1^-)=\infty$, $g_1(0^+)=bR(a,b)$.

b) Let $g_{21}(r)=\mu_{p,q}(r)-\pi_{p,q}(r)/2$, $g_{22}(r)=\log(r'/r)$, then $g_{21}(1/2^{1/q})=g_{22}(1/2^{1/q})=0$. By differentiation, we have

$$g_{21}'(r)=\frac{-\pi_{p,q}^2}{4rr'^q\kappa_{p,q}^2},g_{22}'(r)=-\frac{1}{rr'^q},\frac{g_{21}'(r)}{g_{22}'(r)}=\frac{\pi_{p,q}^2}{4\kappa_{p,q}^2}.$$

which is decreasing. Hence the monotonicity of $g_2(r)$ follows and the limiting values are clear. The second inequality follows from the identity $\mu_{p,q}(r)\mu'_{p,q}(r)=\pi_{p,q}^2/4$

c) We write $g_3(r)$ as $g_{31}(r)/g_{32}(r)$, where $g_{31}(r)=\log(1/(1-r^{q/2}))$ and $g_{32}(r)=1/\mu_{p,q}(r)$, then $g_{31}(0)=g_{32}(0)=0$. After simplifications, we get

$$\frac{g_{31}'(r)}{g_{32}'(r)}=\frac{q(1+r^{q/2})r^{q/2}\kappa_{p,q}^2(r')}{2},$$

which is increasing from $(0,1)$ onto $(0,q\pi_{p,q}^2/4)$ by Lemma 4 (a) and the limiting values are clear.

**Remark** For $p=q=2$. Theorem 2 reduces to Theorem 1.4 in [11], Exercise 5.68(37)−(39) in [3].

### 3.3　定理 3 的证明

### 3.3　Proof of Theorem 3

a) Let $x=1/r$, then $\mathrm{d}x/\mathrm{d}r=-x^2$ and

$$G_1'(r)=\frac{-\pi_{p,q}^2}{4xx'^q\kappa_{p,q}^2(x)}\frac{-1}{r^2}=\frac{\pi_{p,q}^2 x}{4x'^q\kappa_{p,q}^2(x)},$$

which is positive and strictly increasing in $x$, so decreasing in $r$. Hence, we get the monotonicity and concavity of $G_1(r)$, and the limiting values are clear. By the concavity of $G_1(r)$, for $s$, $t\in(1,\infty)$ and $\lambda\in(0,1)$, we have

$$G_1(\lambda s+(1-\lambda)t)\geqslant\lambda G_1(s)+(1-\lambda)G_1(t).$$

Let $x=1/s$, $y=1/t$, then we have

$$\lambda\mu_{p,q}(x)+(1-\lambda)\mu_{p,q}(y)\leqslant\mu_{p,q}\left(\frac{xy}{\lambda y+(1-\lambda)x}\right),$$

with equality if and only if $x=y$.

b) Let $x=rt/(1+t')$, then $x<rt<r$ and $\mathrm{d}x/\mathrm{d}r=x/r$,

$$G_2'(r)=\frac{\pi_{p,q}^2(x'^q\kappa_{p,q}^2(x)-r'^q\kappa_{p,q}^2(r))}{4r\,(x'^{q/2}\kappa_{p,q}(x)r'^{q/2}\kappa_{p,q}(r))^2}$$

which is positive by Lemma 4(a), then the monotonicity of $G_2(r)$ follows. Clearly, the limiting values are

$$G_2(1^-)=\mu_{p,q}(t/(1+t')),$$

$$G_2(0^+)=\lim_{r\to 0^+}\{(\mu_{p,q}(x)+\log x)-(\mu_{p,q}(r)+\log r)+\log((1+t')/t)\}=\mathrm{arth}\,t'.$$

c) Let $x=\sqrt{r}$, then $x<r$, $\mathrm{d}x/\mathrm{d}r=1/(2x)$, we have the derivative

$$G_3'(r)=\frac{\pi_{p,q}^2(m_{p,q}(r)-2m_{p,q}(x))}{8r\,(x'^{q/2}\kappa_{p,q}(x')r'^{q/2}\kappa_{p,q}(r))^2}$$

which is negative by Lemma 5(d). By Ramanujan's asymptotic formula[17] $B(a_1,b_1)F(a_1,b_1,c_1;r)+\log(1-r)=R(a_1,b_1)+O((1-r)\log(1-r))$ as $r\to 1^-$, then we have

$$G_3(0^+)=\lim_{r\to 0^+}\frac{F(a,b,c;r'^q)}{F(a,b,c;1-r^{q/2})}=\lim_{r\to 0^+}\frac{\log r^q}{\log r^{q/2}}=2,$$

$$G_3(1^-)=\lim_{r\to 1^-}\frac{F(a,b,c;r^{q/2})}{F(a,b,c;r^q)}=\lim_{r\to 1^-}\frac{\log(1-r^{q/2})}{\log(1-r^q)}=1.$$

### 3.4 定理4的证明

### 3.4 Proof of Theorem 4

a) We write $H_1(r)$ as

$$H_1(r)=\frac{\pi_{p,q}}{2}\frac{r'^qF(a,b,c;r'^q)}{\log(1/r^q)}\frac{\log(1/r'^q)}{r^qF(a,b,c;r^q)}.$$

Hence, Theorem 4(a) follows from Lemma 6(c).

b) We write $H_2(r)$ as

$$H_2(r)=\frac{\pi_{p,q}}{2}\frac{\mathrm{arth}\,r^{q/2}}{r^{q/2}\kappa_{p,q}(r)}\frac{r'^{q/2}\kappa_{p,q}'(r)}{\mathrm{arth}\,r'^{q/2}}.$$

Hence, Theorem 4(b) follows from Lemma 4(e).

## 参考文献:

## References:


[1] Kamiya T, Takeuchi S. Complete $(p,q)$-elliptic integrals with application to a family of means[J]. Journal of Classical Analysis, 2017, 10(1): 15-25.

[2] Takeuchi S. Legendre-type relations for generalized complete elliptic integrals[J]. Journal of Classical Analysis, 2016, 9(1): 35-42.

[3] Anderson G D, Vamanamurthy M K, Vuorinen M. Conformal Invariants, Inequalities, and Quasiconformal Maps[M].

New York：John Wiley and Sons，1997：48-73.

[4] Anderson G D，Vamanamurthy M K，Vuorinen M. Functional inequalities for complete elliptic integrals and their ratios [J]. SIAM Journal on Mathematics Analysis，1990，21(2)：536-549.

[5] Alzer H，Qiu S L. Monotonicity theorems and inequalities for the complete elliptic integrals[J]. Journal of Computational and Applied Mathematics，2004，172(2)：289-312.

[6] Borwein J M，Borwein P B. Pi and the AGM [M]. New York：A Wiley-Interscience Publication，1987：1-32.

[7] Qiu S L，Vamanamurthy M K，Vuorinen M. Some inequalities for the growth of elliptic integrals[J]. SIAM Journal on Mathematics Analysis，1998，29(5)：1224-1237.

[8] Takeuchi S. A new form of the generalized complete elliptic integrals[J]. Kodai Mathematical Journal，2016，39(1)：202-226.

[9] Lehto O，Virtanen K I. Quasiconformal Mappings in the Plane [M]. New York：Springer-Verlag，1973：59-67.

[10] Berndt B C. Ramanujan′s Notebooks：Part Ⅰ[M]. New York：Springer-Verlag，1985：357.

[11] QIU S L，Vuorinen M. Infinite products and normalized quotients of hypergeometric functions[J]. SIAM Journal on Mathematical Analysis，1999，30(5)：1057-1075.

[12] Qiu S L，Vuorinen M. Duplication inequalities for the ratios of hypergeometric functions[J]. Forum Mathematicum，2000，12(1)：109-133.

[13] Ponnusamy S，Vuorinen M. Asymptotic expansions and inequalities for hypergeometric function[J]. Mathematika，1997，44(2)：278-301.

[14] 焦仁兵，裘松良，葛耿韬. 广义 $(p,q)$-椭圆积分的单调性和凹凸性[J]. 浙江理工大学学报(自然科学版)，2018，39(6)：765-769.

[15] 裘松良，丁志栓，王婕. $(p,q)$-Grötzsch 环函数与 $(p,q)$-Hübner 函数的一些性质 [J]. 浙江理工大学学报(自然科学版)，2020，43(6)：846-851.

[16] Anderson G D，Barnard R W，Richards K C，et al. Inequalities for zero-balanced hypergeometric functions [J]. Transactions of the American Mathematical Society，1995，347(5)：1713-1723.

[17] Berndt B C. Ramanujan's Notebook：Part Ⅱ[M]. New York：Springer-Verlag，Berlin，Heidelberg，1989：71.

(责任编辑:康　锋)