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Sharp inequalities for the scale invariant Cassinian metric

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Abstract: In this paper, the authors study sharp inequalities between the scale invariant Cassinian metric and some hyperbolic type metrics. They also prove sharp distortion inequalities of the scale invariant Cassinian metric under Möbius transformations from the unit ball onto itself or from the upper half space onto itself.

Key words: the scale invariant Cassinian metric; hyperbolic type metrics; Möbius transformations

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伸缩不变 Cassini 度量的精确不等式

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摘　要：研究了伸缩不变 Cassini 度量与一些双曲型度量的精确比较关系，还证明了伸缩不变 Cassini 度量在单位球映到自身或半空间映到自身的 Möbius 变换下的精确偏差不等式。

关键词：伸缩不变 Cassini 度量；双曲型度量；Möbius 变换

0　Introduction

Due to Liouville's theorem, in the higher dimensional Euclidean spaces the hyperbolic metric can only be defined in balls and half spaces. As generalizations of the hyperbolic metric, hyperbolic type metrics defined in general domains in the higher dimensional Euclidean spaces play important roles in the study of geometric function theory. Compared with the hyperbolic metric, they have the advantages of being easy to calculate and estimate. The comparison between hyperbolic type metrics and the distortion properties of these metrics under Möbius transformations are two main themes of this research.

Recently, the scale invariant Cassinian metric was introduced by Ibragimov in [1]. Some basic properties and distortion inequalities of the scale invariant Cassinian metric under Möbius transformations were investigated in [1-3]. In [1, 4], the authors studied the comparison between the scale invariant Cassinian metric and some hyperbolic type metrics, while some statements about the sharpness of comparison are missing.

In this paper, we continue the research of the scale invariant Cassinian metric. We study sharp inequalities between the scale invariant Cassinian metric and the absolute ratio metric, the half-Apollonian

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metric, the Apollonian metric or the Cassinian metric, in the sense that there exists a domain such that the equality holds for some pair of points in the domain. We also prove sharp distortion inequalities of the scale invariant Cassinian metric under some families of Möbius transformations.

1 Preliminaries

In this section, we recall the definitions of the hyperbolic metric and some hyperbolic type metrics.

For a domain $D \subsetneq \mathbf{R}^n$ and $x, y \in D$, the scale invariant Cassinian metric is defined by

$$\tilde{\tau}_D(x,y) = \log\left(1 + \sup_{p \in \partial D} \frac{|x-y|}{\sqrt{|x-p||p-y|}}\right).$$

The hyperbolic metrics ρ_{B^n} and ρ_{H^n} of the unit ball $B^n = \{z \in \mathbf{R}^n : |z| < 1\}$ and of the upper half space $H^n = \{(x_1, \cdots, x_n) \in \mathbf{R}^n : x_n > 0\}$ are defined as follows. By [5, p.40] we have for $x, y \in B^n$,

$$\operatorname{sh}\frac{\rho_{B^n}(x,y)}{2} = \frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}},$$

and by [5, p.35] for $x, y \in H^n$,

$$\operatorname{ch}\rho_{H^n}(x,y) = 1 + \frac{|x-y|^2}{2x_n y_n}.$$

It is well known that the hyperbolic metric provides a powerful tool in complex analysis and geometric function theory, and many theorems have more natural explanations under the hyperbolic metric than the Euclidean metric.

The following lemma shows the relation between the scale invariant Cassinian metric and the hyperbolic metric.

Lemma 1 [1, Theorem 3.8; 3, Theorem 4.1, Theorem 4.4] Let $D = B^n$ or $D = H^n$. For all $x, y \in D$, we have

$$\frac{1}{4}\rho_D(x,y) \leqslant \tilde{\tau}_D(x,y) \leqslant \rho_D(x,y).$$

Both inequalities are sharp.

Let D be an open subset of $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ with card $\partial D \geqslant 2$. The absolute ratio metric δ_D is defined as (see [6])

$$\delta_D(x,y) = \log(1 + \sup_{a,b \in \partial D} |a,x,b,y|)$$

for all $x, y \in D$, where

$$|a,x,b,y| = \frac{|a-b||x-y|}{|a-x||b-y|} \text{ with } \frac{|\infty - b|}{|\infty - x|} = 1$$

is the absolute ratio.

Let D be a proper open subset of $\overline{\mathbf{R}}^n$ and for all $x, y \in D$, the Apollonian metric α_D is defined as

$$\alpha_D(x,y) = \sup_{a,b \in \partial D} \log |a,x,y,b|.$$

Note that α_D is a pseudo-metric in D. It is, in fact, a metric if and only if $\overline{\mathbf{R}}^n \backslash D$ is not contained in an $(n-1)$-dimensional sphere in $\overline{\mathbf{R}}^n$, see [7, Theorem 1.1].

The absolute ratio metric and the Apollonian metric coincide with the hyperbolic metric if D is the unit ball B^n or the upper half space H^n, see [8, Lemma 8.39] and [7, Example 3.2, Lemma 3.1]. It is easy to see that both the absolute ratio metric and the Apollonian metric are Möbius invariant. They are useful in the study of uniform domains.

In [9], Hästö and Lindén gave another form of the Apollonian metric:

$$\alpha_D(x,y) = \sup_{a \in \partial D} \log \frac{|a-y|}{|a-x|} + \sup_{b \in \partial D} \log \frac{|b-x|}{|b-y|} \tag{1}$$

Using only one term in the right-hand side of (1), Hästö defined the half-Apollonian metric as follows.

Let D be a proper open subset of $\mathbf{R}^n$ and for all $x, y \in D$, the half-Apollonian metric η_D is defined as (see [9])

$$\eta_D(x, y) = \sup_{p \in \partial D} \left| \log \frac{|x-p|}{|y-p|} \right|.$$

Note that η_D is also a pseudo-metric in D, and a proper metric whenever $\mathbf{R}^n \setminus D$ is not a subset of a hyperplane (see [9, Theorem 1.2]). By [7, Lemma 2.2 (i)],

$$\alpha_D(x, y) = \eta_D(x, y)$$

where $D = \overline{\mathbf{R}}^n \setminus \{\zeta, \infty\}$ for any $\zeta \in \mathbf{R}^n$.

The half-Apollonian metric is bilipschitz equivalent to the Apollonian metric.

Lemma 2 [9, Theorem 2.1] Let $D \subsetneq \mathbf{R}^n$ be a domain. Then the double inequality

$$\frac{1}{2}\alpha_D(x, y) \leqslant \eta_D(x, y) \leqslant \alpha_D(x, y)$$

holds for all $x, y \in D$. Both inequalities are sharp.

Let D be a proper subdomain of $\overline{\mathbf{R}}^n$ and for all $x, y \in D$, the Cassinian metric c_D is defined as (see [10])

$$c_D(x, y) = \sup_{p \in \partial D} \frac{|x-y|}{|x-p||y-p|}.$$

The Cassinian metric and the hyperbolic metric in the unit ball are related by [11, Corollary 3.3]:

$$c_{B^n}(x, y) \geqslant \frac{1}{2}\rho_{B^n}(x, y).$$

2 The $\tilde{\tau}$-metric and some hyperbolic type metrics

In this section, we compare the $\tilde{\tau}_D$-metric with the absolute ratio metric δ_D, the half-Apollonian metric η_D, the Apollonian metric α_D, and the Cassinian metric c_D, respectively. In particular, the sharp inequalities between the $\tilde{\tau}_D$-metric and these hyperbolic type metrics are studied.

Theorem 1 Let $D \subsetneq \mathbf{R}^n$ be a domain. Then the double inequality

$$\frac{1}{4}\delta_D(x, y) \leqslant \tilde{\tau}_D(x, y) \leqslant \delta_D(x, y) \tag{2}$$

holds for all $x, y \in D$. Both inequalities are the best possible.

Proof. The double inequality and the sharpness of the right－hand side of (2) are the facts of [4, Theorem 5.2].

For the sharpness of the left-hand side of inequalities (2), we consider the domain $D = B^n$, then $\delta_{B^n} = \rho_{B^n}$. By Lemma 1, the constant $\frac{1}{4}$ is the best possible.

This completes the proof.□

Theorem 2 Let $D \subsetneq \mathbf{R}^n$ be a domain with $\partial D \neq \emptyset$. Then the double inequality

$$\frac{1}{2}\eta_D(x, y) \leqslant \tilde{\tau}_D(x, y) \leqslant \frac{1}{2}\eta_D(x, y) + \log 3 \tag{3}$$

holds for all $x, y \in D$. The constant $\frac{1}{2}$ in the left-hand side and the constant log3 in the right-hand side of the inequalities are the best possible.

Proof. The double inequality is the fact of [1, Theorem 3.5].

For the sharpness of the constant $\frac{1}{2}$ in the left-hand side and the sharpness of the constant log3 in the right-hand side of inequalities (3), we consider the domain $D=\mathbf{R}^n\setminus\{e_1\}$. Let $y=-x=te_1$ with $t>1$. By [2, Proposition 3.1], we have

$$\tilde{\tau}_D(x,y)=\log\left(1+\frac{2t}{\sqrt{t^2-1}}\right)\text{ and }\eta_D(x,y)=\log\frac{t+1}{t-1}.$$

We further obtain

$$\lim_{t\to1^+}\frac{\tilde{\tau}_D(x,y)}{\eta_D(x,y)}=\lim_{t\to1^+}\frac{\log\left(1+\frac{2t}{\sqrt{t^2-1}}\right)}{\log\left(1+\frac{2}{t-1}\right)}=\frac{1}{2}$$

and

$$\lim_{t\to\infty}\left(\tilde{\tau}_D(x,y)-\frac{1}{2}\eta_D(x,y)\right)=\lim_{t\to\infty}\log\left(\frac{2t+\sqrt{t^2-1}}{\sqrt{t^2-1}}\cdot\sqrt{\frac{t-1}{t+1}}\right)=\log3.$$

This completes the proof.□

Theorem 3 Let $D\subsetneq\mathbf{R}^n$ be a domain. Then the double inequality

$$\frac{1}{4}\alpha_D(x,y)\leqslant\tilde{\tau}_D(x,y)\leqslant\frac{1}{2}\alpha_D(x,y)+\log3 \tag{4}$$

holds for all $x,y\in D$. The constant $\frac{1}{4}$ in the left-hand side and the constant log3 in the right-hand side of the inequalities are the best possible.

Proof. The double inequality follows easily from Theorem 2 and Lemma 2.

For the sharpness of the constant $\frac{1}{4}$ in the left-hand side of inequalities (4), we consider the domain $D=H^n$, then $\alpha_{H^n}=\rho_{H^n}$. By Lemma 1, the constant $\frac{1}{4}$ is the best possible.

For the sharpness of the constant log3 in the right-hand side of inequalities (4), we consider the domain $D=\overline{\mathbf{R}}^n\setminus\{e_1,\infty\}$, then $\alpha_D=\eta_D$. By Theorem 2, the constant log3 is the best possible.

This completes the proof.□

Theorem 4 For every domain $D\subsetneq\mathbf{R}^n$ the inequality

$$\tilde{\tau}_D(x,y)\geqslant\log(1+rc_D(x,y))$$

holds for all $x,y\in D$, where $r=\min\{d(x),d(y)\}$ and $d(x)$ is the distance from x to the boundary of D. The inequality is sharp.

Proof. Let $p\in\partial D$ such that

$$c_D(x,y)=\frac{|x-y|}{|x-p||p-y|}.$$

Then

$$\begin{aligned}\tilde{\tau}_D(x,y)&=\log\left(1+\frac{|x-y|}{\sqrt{|x-p||p-y|}}\right)\\&=\log(1+\sqrt{|x-p||p-y|}\,c_D(x,y))\\&\geqslant\log(1+rc_D(x,y)).\end{aligned}$$

For the sharpness, we consider the punctured space $D_p=\mathbf{R}^n\setminus\{p\}$. Let $x,y\in D_p$ with $|x-p|=|y-p|$. It is clear that

$$\tilde{\tau}_D(x,y)=\log\left(1+\frac{|x-y|}{|x-p|}\right)=\log(1+|y-p|\,c_D(x,y)).$$

Hence the inequality is sharp.□

3 The $\tilde{\tau}$-metric and Möbius transformations

In [2, Proposition 3.1], Mohapatra and Sahoo gave a formula for the $\tilde{\tau}_{B^n}$-metric in the special case when $x = ty\,(x, y \in B^n)$ with $t \in \mathbf{R}\setminus\{0\}$. In [3], the authors studied the geometry of the $\tilde{\tau}_{B^n}$-metric and obtained formulas for this metric in more special cases, some of which will be used in the proof of the theorems in this section. They also proved that $\tilde{\tau}_D$-metric is not changed by more than a factor 2 under Möbius transformations, see [3, Theorem 5.1].

In [1], Ibragimov showed the following distortion inequalities of $\tilde{\tau}_{B^n}$-metric.

Theorem 5 [1, Theorem 4.2] Let f be a Möbius transformation of B^n. Then

$$\frac{1}{2}\tilde{\tau}_{B^n}(x,y) - \log\frac{5}{4} \leqslant \tilde{\tau}_{B^n}(f(x), f(y)) \leqslant 2\tilde{\tau}_{B^n}(x,y) + \log\frac{5}{4}$$

for all $x, y \in B^n$.

The following theorem improves the result in Theorem 5 when f is an inversion in some sphere $S^{n-1}(a^*, r)$ with center a^* and radius r.

Theorem 6 For $a \in B^n\setminus\{0\}$. Let $a^* = \dfrac{a}{|a|^2}$, $r = \sqrt{|a^*|^2 - 1}$ and $f(z) = a^* + r^2\dfrac{(z - a^*)}{|z - a^*|^2}$ be the inversion in $S^{n-1}(a^*, r)$. Then $f(B^n) = B^n$ and for all $x, y \in B^n$,

$$\frac{1}{2}\tilde{\tau}_{B^n}(x,y) \leqslant \tilde{\tau}_{B^n}(f(x), f(y)) \leqslant 2\tilde{\tau}_{B^n}(x,y).$$

Both inequalities are the best possible.

Proof. The double inequality is clear by [3, Theorem 5.1].

For the sharpness of the right-hand side of the inequalities, let $a = te_1$ with $\dfrac{1}{2} < t < 1$. Then $r = \dfrac{\sqrt{1-t^2}}{t}$. Putting $x = (1-t)e_1$ and $y = -(1-t)e_1$, we have

$$f(x) = \frac{2t-1}{1-t+t^2}e_1 \text{ and } f(y) = \frac{1}{1+t-t^2}e_1.$$

By [2, Proposition 3.1], we have

$$\lim_{t\to 1^-}\frac{\tilde{\tau}_{B^n}(f(x), f(y))}{\tilde{\tau}_{B^n}(x,y)} = \lim_{t\to 1^-}\frac{\log\left(1 + \dfrac{2(1-t)(1+t)}{\sqrt{t(2-t)(1-t+t^2)(1+t-t^2)}}\right)}{\log\left(1 + \dfrac{2(1-t)}{\sqrt{t(2-t)}}\right)} = 2.$$

Thus the constant 2 is attained. The sharpness of the left－hand side of the inequalities can be seen by considering the inverse of f and hence the constant $\dfrac{1}{2}$ is also the best possible.

This completes the proof.□

Theorem 7 Let $f(z) = a + r^2\dfrac{(z-a)}{|z-a|^2}$ be the inversion in $S^{n-1}(a, r)$ with $\mathrm{Im}a = 0$. Then $f(H^n) = H^n$ and for all $x, y \in H^n$,

$$\frac{1}{2}\tilde{\tau}_{H^n}(x,y) \leqslant \tilde{\tau}_{H^n}(f(x), f(y)) \leqslant 2\tilde{\tau}_{H^n}(x,y).$$

Both inequalities are the best possible.

Proof. The double inequality is clear by [3, Theorem 5.1].

For the sharpness of the left-hand side of the inequalities, let $a = 0$ and $r = 1$. Put $x = e_1 + te_2$ and $y = e_1 + \dfrac{1}{t}e_2$ with $0 < t < \sqrt{2} - 1$, then

$$f(x)=\frac{1}{1+t^2}e_1+\frac{t}{1+t^2}e_2 \text{ and } f(y)=\frac{t^2}{1+t^2}e_1+\frac{t}{1+t^2}e_2.$$

By [3, Lemma 3.11 (1), Lemma 3.12], we have

$$\lim_{t\to 0^+}\frac{\tilde{\tau}_{H^n}(f(x),f(y))}{\tilde{\tau}_{H^n}(x,y)}=\lim_{t\to 0^+}\frac{\log\left(1+\sqrt{\frac{1-t^2}{t}}\right)}{\log\left(1+\frac{1}{t}-t\right)}=\frac{1}{2}.$$

Thus the constant $\frac{1}{2}$ is attained. The sharpness of the right-hand side of the inequalities can be seen by considering the inverse of f and hence the constant 2 is also the best possible.

This completes the proof.□

4 Concluding Remark

There are several hyperbolic type metrics which preserve some characterizations of the hyperbolic metric and hence are useful in the geometric function theory. Different hyperbolic type metrics have their own interests in the applications to specific problems. There are close relationships between hyperbolic type metrics as the results of the present paper show. The sharp inequalities obtained in this paper are expected to be helpful in the study of the metric ball inclusion problems.

References:

[1] Ibragimov Z. A scale-invariant Cassinian metric[J]. The Journal of Analysis, 2016, 24(1): 111-129.

[2] Mohapatra M R, Sahoo S K. Mapping properties of a scale invariant Cassinian metric and a Gromov hyperbolic metric[J]. Bulletin of the Australian Mathematical Society, 2018, 97(1): 141-152.

[3] Wang G D, Xu X X, Vuorinen M. Remarks on the scale invariant Cassinian metric[EB/OL]. (2019-03-21)[2019-06-26]. https://arxiv.org/abs/1903.09099.

[4] Mohapatra M R, Sahoo S K. A Gromov hyperbolic metric vs the hyperbolic and other related metrics[J]. Computational Methods and Function Theory, 2018, 18(3): 473-493.

[5] Beardon A F. The geometry of discrete groups[M]. New York: Springer-Verlag, 1983.

[6] Seittenranta P. Möbius-invariant metrics[J]. Mathematical Proceedings of the Cambridge Philosophical Society, 1999, 125(3): 511-533.

[7] Beardon A F. The Apollonian metric of a domain in $\mathbf{R}^n$ [M]//Quasiconformal Mappings and Analysis. New York, NY: Springer New York, 1998: 91-108.

[8] Vuorinen M. Conformal Geometry and Quasiregular Mappings[M]. Berlin: Springer, 1988.

[9] Hästö P, Lindén H. Isometries of the half-Apollonian metric[J]. Complex Variables, Theory and Application: an International Journal, 2004, 49(6): 405-415.

[10] Ibragimov Z. The Cassinian metric of a domain in $\overline{\mathbf{R}}^n$ [J]. Uzbekskiĭ Matematicheskiĭ Zhurnal, 2009(1): 53-67.

[11] Ibragimov Z, Mohapatra M R, Sahoo S K, et al. Geometry of the Cassinian metric and its inner metric[J]. Bulletin of the Malaysian Mathematical Sciences Society, 2017, 40(1): 361-372.

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