浙江理工大学学报，2019，41(6)：823-828
Journal of Zhejiang Sci-Tech University
DOI：10.3969/j.issn.1673－3851(n).2019.06.018

Some properties of zero-balanced hypergeometric functions

MA Hanxi，*QIU Songliang*，*BAO Qi*
(School of Sciences，Zhejiang Sci-Tech University，Hangzhou 310018，China)

Abstract：In this paper，the authors obtain several monotonicity properties and sharp inequalities for the zero-balanced hypergeometric function $F(a,b;a+b;x)$，by studying the analytic properties of certain combinations defined in terms of $F(a,b;a+b;x)$ and some elementary functions such as trigonometric functions，thus extending several known related results for the complete elliptic integrals of the first kind to zero-balanced hypergeometric functions.

Key words：zero-balanced hypergeometric function；monotonicity；absolute monotonicity；inequality
CLC number：O174.6　　Document code：A　　Article ID：1673-3851 (2019) 11-0823-06

零平衡超几何函数的几个性质

马晗茜，裘松良，鲍　琪
(浙江理工大学理学院，杭州 310018)

摘　要：通过研究、揭示由零平衡超几何函数 $F(a,b;a+b;x)$ 与三角函数等初等函数的适当组合的分析性质，获得了 $F(a,b;a+b;x)$ 的单调性、绝对单调性和由初等函数给出的上下界等性质，从而将完全椭圆积分的相关已知结果推广到零平衡超几何函数。

关键词：零平衡超几何函数；单调性；绝对单调性；不等式

0　Main Results

For real numbers $a,b,c(c\neq 0,-1,-2,\cdots)$ the Gaussian hypergeometric function is defined by

$$F(a,b;c;x)={}_2F_1(a,b;c;x)\equiv\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!},\ |x|<1 \tag{1}$$

where $(a,0)=1$ for $a\neq 0$, and for $n=1,2,\cdots$, $(a,n)=a(a+1)\cdots(a+n-1)$ is the shifted factorial function. The function $F(a,b;c;x)$ is said to be zero-balanced if $c=a+b$. It is well known that $F(a,b;c;x)$ has wide important applications in many fields of mathematics, and in physics and engineering as well [1-4]. For the properties and applications of the hypergeometric functions, the readers are referred to [1, 3, 5-8].

It is also well known that many other special functions in mathematical physics are particular or limiting cases of $F(a,b;c;x)$. For example, the generalized complete elliptic integrals of the first kind are defined as

Received date：2019－05－21　　Published Online：2019－09－02
Fund item：This research is supported by the NSF of P.R. China (Grant No.11771400).
Introduction to the first author：MA Hanxi (1996－　)，female，Wenzhou，Zhejiang，postgraduate，research interests：quasi-conformal theory，special functions，Ramanujan's modular equations.
Corresponding author：QIU Songliang，E-mail：sl_qiu@zstu.edu.cn

$$\mathrm{K}_a=\mathrm{K}_a(r)=\frac{\pi}{2}F(a,1-a;1;r^2),\mathrm{K}_a{}'=\mathrm{K}'_a(r)=\mathrm{K}_a(r'),$$

for $r\in(0,1)$, $r'=\sqrt{1-r^2}$ and for $a\in(0,1/2]$, while $\mathrm{K}=\mathrm{K}(r)=\mathrm{K}_{1/2}(r)$ and $\mathrm{K}'=\mathrm{K}'(r)=\mathrm{K}'_{1/2}(r)$ are the well-known complete elliptic integrals of the first kind (cf. [1, 2, 9-10]). Clearly, $\mathrm{K}_a(0)=\pi/2$ and $\mathrm{K}_a(1)=\infty$.

For real numbers $x,y\in(0,\infty)$, the gamma, beta and psi (digamma) functions are defined by

$$\begin{cases}\Gamma(x)=\int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t\\ B(x,y)=\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\\ \psi(x)=\dfrac{\Gamma'(x)}{\Gamma(x)}\end{cases}\tag{2}$$

respectively (cf. [1, 4]). Clearly, $(a,n)=\Gamma(n+a)/\Gamma(a)$. Let

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)=0.577125664\cdots,$$

be the Euler-Mascheroni constant, and for $a,b\in(0,\infty)$, let

$$R(a,b)=-2\gamma-\psi(a)-\psi(b)\tag{3}$$

which is called the Ramanujan constant or Ramanujan R-function and plays an important role in the studies of hypergeometric and quasiconformal special functions (cf. [7-11]). Clearly, $R(1/2,1/2)=\log16$.

Throughout this paper, for convenience, we let

$$B=B(a,b),R=R(a,b),B_+=B(a+1,b+1),R_+=R(a+1,b+1),\alpha=ab/(a+b),$$
$$F(x)=F(a,b;a+b;x),G(x)=F(a,b;a+b+1;x),F_+(x)=F(a+1,b+1;a+b+2;x).$$

By (2) and [1, 6.1.15 & 6.3.5],

$$\begin{cases}B_+=\dfrac{\alpha B}{a+b+1}\\ R_+=R-\dfrac{1}{\alpha}\end{cases}\tag{4}$$

During the past few decades, many authors obtained various properties for K and K_a, some of which have been extended to the hypergeometric functions (cf. [3, 5-10, 12-15]). Among these results are the followings:

a) In [6, Theorem 3.2(3)], it states that the function $r\mapsto r^2\mathrm{K}(r)+\lambda\log r'$ is increasing (decreasing) on $(0,1)$ if and only if $\lambda\leqslant1$ ($\lambda\geqslant\pi$, respectively).

b) In [12, Theorem 2], it was proved that the functions $r\mapsto\frac{1}{r'}\sin(r'\mathrm{K}(r))$ and $r\mapsto\frac{1}{r^2}\cos(r'\mathrm{K}(r))$ are both strictly increasing on $(0,1)$ with ranges $(1,\infty)$ and $(\pi/8,1)$, respectively.

Based on these known results above-mentioned, it is natural to ask whether [6, Theorem 3.2(3)] and [12, Theorem 2] can be extended to the zero-balanced hypergeometric functions. For instance, for what values of $\lambda\in(-\infty,\infty)$, the function $x\mapsto BxF(a,b;a+b;x)+\lambda\log(1-x)$ is monotone on $(0,1)$?The purpose of this paper is to give an affirmative answer to this question, by proving the following main results.

Theorem 1 For $a,b\in(0,\infty)$ with $c=a+b$, and for each number $\lambda\in(-\infty,\infty)$, let $f_1(x)=BxF(a,b;a+b;x)+\lambda\log(1-x)$ and $f_2(x)=f_1(x)/x$. Then we have the following conclusions:

a) $f_1(0)=0$, $f_1(1^-)=R$ if $\lambda=1$, $f_1(1^-)=\infty$ if $\lambda<1$, and $f_1(1^-)=-\infty$ if $\lambda>1$.

b) If $a\leqslant\min\{1,c/2\}$, then the function f_1 is increasing (decreasing) on $(0,1)$ if and only if $\lambda\leqslant1$ ($\lambda\geqslant B$, respectively). If $1<\lambda<B$, then there exists a number $r_1\in(0,1)$ such that f_1 is increasing on (0,

$r_1]$ and decreasing on $[r_1,1)$. Moreover, $f_1(-f_1)$ is absolutely monotone on $(0,1)$ if $\lambda\leqslant 1$ ($\lambda\geqslant B$, respectively).

c) If $ab\geqslant\max\{1,c/2\}$, then f_1 is increasing (decreasing) if and only if $\lambda\leqslant B$ ($\lambda\geqslant 1$, respectively). If $B<\lambda<1$, then there exists a number $r_2\in(0,1)$ such that f_1 is decreasing on $(0,r_2]$ and increasing on $[r_2,1)$. Moreover, $f_1(-f_1)$ is absolutely monotone on $(0,1)$ if $\lambda\leqslant B$ ($\lambda\geqslant 1$, respectively).

d) In the case when $ab\leqslant\min\{3-c,(c+1)/3\}$, if $\lambda\leqslant\min\{ab,2\alpha B\}$ ($\lambda\geqslant 2\alpha B$), then the function f_2 is increasing (decreasing) from $(0,1)$ onto $(B-\lambda,f_1(1^-))$ ($(f_1(1^-),B-\lambda)$, respectively).

e) In the case when $ab\geqslant\max\{3-c,(c+1)/3\}$, if $\lambda\leqslant 2\alpha B$ ($\lambda\geqslant\max\{ab,2\alpha B\}$), then f_2 is increasing (decreasing) from $(0,1)$ onto $(B-\lambda,f_1(1^-))$ ($(f_1(1^-),B-\lambda)$, respectively).

Theorem 2 Let $c=a+b$ for $a,b\in(0,\infty)$.

a) If $\alpha\leqslant 1/4$, then the function $g_1(x)\equiv(1-x)^{-1/2}\sin(\pi(1-x)^{2\alpha}F(x)/2)$ is strictly increasing from $[0,1)$ onto $[1,\infty)$. In particular, for each $a\in(0,1/2]$, the function $g_2(r)\equiv\frac{1}{r'}\sin(r'\mathrm{K}_a(r))$ is strictly increasing from $[0,1)$ onto $[1,\infty)$.

b) For each $\beta\in(-\infty,\infty)$, the function $g_3(x)\equiv(1-x)^{-\beta}\sin(\pi(1-x)^{\alpha}F(x)/2)$ is increasing (decreasing) from $[0,1)$ onto $[1,\infty)$ ($(0,1)$) if and only if $\beta\geqslant\alpha$ ($\beta\leqslant 0$, respectively). If $0<\beta<\alpha$, then there exists a number $r_3\in(0,1)$ such that g_3 is increasing on $(0,r_3]$ and decreasing on $[r_3,1)$.

c) If $\alpha\leqslant 1/3$, then the function $g_4(x)\equiv\frac{1}{x}\cos(\pi(1-x)^{2\alpha}F(x)/2)$ is strictly increasing from $(0,1)$ onto $(\alpha\pi/2,1)$. In particular, for each $a\in(0,1/2]$, the function $g_5(x)\equiv r^{-2}\cos(r'\mathrm{K}_a(r))$ is strictly increasing from $(0,1)$ onto $(\pi a(1-a)/2,1)$. Moreover, for $a,b\in(0,\infty)$ with $\alpha\leqslant 1/3$ and for all $x\in(0,1)$,

$$\frac{2\arccos x}{\pi(1-x)^{2\alpha}}<F(a,b;a+b;x)<\frac{2\arccos(\alpha\pi x/2)}{\pi(1-x)^{2\alpha}}\tag{5}$$

and for $a\in(0,1/2]$, $r\in[0,1)$, and $\rho=a(1-a)$,

$$\frac{\arccos r^2}{(1-r^2)^{2\rho}}<\mathrm{K}_a(r)<\frac{\arccos(\pi\rho r^2/2)}{(1-r^2)^{2\rho}}\tag{6}$$

1 Preliminaries

In this section, we prove two lemmas needed in the proofs of our main results stated in Section 0.

First, let us recall the following concept: An infinitely-differentiable function f defined on an interval I is said to be absolutely monotone on I if f and its derivatives of all orders are nonnegative at all points in I. Such kind of functions were first investigated by S. N. Bernstein (see [16]).

Next, we recall the following well-known formulas [1]:

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},a+b<c\tag{7}$$

$$\frac{\mathrm{d}F(a,b;c;x)}{\mathrm{d}x}=\frac{ab}{c}F(a+1,b+1;c+1;x)\tag{8}$$

$$F(a,b;c;x)=(1-x)^{c-a-b}F(c-a,c-b;c;x)\tag{9}$$

$$BF(a,b;a+b;x)=R-\log(1-x)+O((1-x)\log(1-x))(x\to 1)\tag{10}$$

We now prove two lemmas.

Lemma 1 For $a,b\in(0,\infty)$ with $c=a+b$, and for $n\in\mathbb{N}_0=\{m\mid m$ is a nonnegative integer$\}$, set

$$a_n=\frac{(n+1)(a,n)(b,n)}{(c,n)n!}\text{ and }b_n=\frac{(n+1)(n+2)(a,n)(b,n)}{(c,n+1)n!}.$$

a) If $ab\leqslant\min\{1,c/2\}$ ($ab\geqslant\max\{1,c/2\}$), then the sequence $\{a_n\}$ is strictly decreasing (increasing, respectively) in $n\in\mathbb{N}_0$, with $a_0=1$ and $a_\infty=\lim_{n\to\infty}a_n=1/B$. In other cases, namely, $c/2<ab<1$ (or

$1 < ab < c/2$), there exists a positive integer $n_1 = n_1(a,b)$ such that $\{a_n\}$ is increasing (decreasing) in $n \leqslant n_1$, and decreasing (increasing, respectively) in $n \geqslant n_1$.

b) If $ab \leqslant \min\{3-c, (c+1)/3\}$ ($ab \geqslant \max\{3-c, (c+1)/3\}$), then the sequence $\{b_n\}$ is strictly decreasing (increasing, respectively) in $n \in \mathbb{N}_0$, with $b_0 = 2/c$ and $b_\infty = \lim_{n\to\infty} b_n = 1/B$. In other cases, that is, $(c+1)/3 < ab < 3-c$ (or $3-c < ab < (c+1)/3$), there exists a positive integer $n_2 = n_2(a,b)$ such that $\{b_n\}$ is increasing (decreasing) in $n \leqslant n_2$, and decreasing (increasing, respectively) in $n \geqslant n_2$.

c) The function $h_1(x) \equiv (\tan x)/x$ is strictly increasing from $(0, \pi/2)$ onto $(1, \infty)$.

Proof. a) Clearly, $a_0 = 1$. By the asymptotic formula of $\Gamma(x)$ [1, 6.1.46],

$$\lim_{n\to\infty} a_n = \frac{1}{B} \lim_{n\to\infty} \frac{(n+1)\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+1)\Gamma(n+c)} = \frac{1}{B}.$$

It is easy to verify that

$$\frac{a_{n+1}}{a_n} = 1 + \frac{(ab-1)n + 2ab - c}{(n+1)^2(n+c)},$$

by which one can easily obtain the assertions on the monotonicity properties of the sequence $\{a_n\}$.

b) Similarly, we have the limiting values $b_0 = 2/c$ and $b_\infty = 1/B$. It is easy to show that

$$\frac{b_{n+1}}{b_n} = 1 + \frac{(ab+c-3)n + 3ab - c - 1}{(n+1)^2(n+c+1)},$$

from which part b) follows.

c) It is well-known, and can been easily proved by applying the Monotone l'Hôpital's Rule [3, Theorem 1.25]. □

Lemma 2 For $a, b \in (0, \infty)$, $c = a + b$, $\mu \in (-\infty, \infty)$ and for $x \in [0, 1)$, let $h_2(x) = F(a, b; c+1; x)/F(a, b; c; x)$ and $h_3(x) = (1-x)^\mu F(a, b; c; x)$.

a) h_2 is strictly decreasing from $(0,1)$ onto $(0,1)$.

b) h_3 is strictly decreasing (increasing) from $(0,1)$ onto $(0,1)$ ($(1,\infty)$) if and only if $\mu \geqslant \alpha$ ($\mu \leqslant 0$, respectively).

Proof. a) The limiting values of h_2 are clear. By (1),

$$h_2(x) = \frac{\sum_{n=0}^{\infty} c_n x^n}{\sum_{n=0}^{\infty} d_n x^n}, c_n = \frac{(a,n)(b,n)}{(c+1,n)n!}, d_n = \frac{(a,n)(b,n)}{(c,n)n!}.$$

Since $c_n/d_n = c/(n+c)$ is strictly decreasing in $n \in \mathbb{N}_0$, the monotonicity property of h_2 follows from [15, Lemma 2.1].

b) It follows from [8, Lemma 2.15(1)] and its proof. □

2 Proofs of the Theorems

2.1 Proof of Theorem 1

Let a_n and b_n be as in Lemma 1, and let $H_1(x) = B(1-x)F(x) + \alpha BxG(x) - \lambda$. By differentiation, (1)-(2) and (8), we obtain

$$f_1'(x) = \frac{H_1(x)}{1-x}, f_1(x) = xf_2(x) = x\sum_{n=0}^{\infty} \frac{Ba_n - \lambda}{n+1} x^n \tag{11}$$

$$H_1'(x) = B\left[2\alpha G(x) - F(x) + \frac{ab\alpha}{c+1} xF_+(x)\right] = B\sum_{n=0}^{\infty} e_n x^n \tag{12}$$

$$x^2(1-x)f_2'(x) = H_2(x) \equiv xH_1(x) - (1-x)f_1(x) \tag{13}$$

$$\frac{1}{x} H_2'(x) = H_3(x) \equiv H_1'(x) + f_2(x) = \sum_{n=0}^{\infty} \frac{abBb_n - \lambda}{n+1} x^n \tag{14}$$

where $e_n = (a,n)(b,n)[(ab-1)n + 2ab - c]/[(c,n+1)n!]$. It is clear that $H_1(0) = B - \lambda$ and $H_1(1^-) =$

$1-\lambda$. By (12), (4) and (10), we obtain

$$H_1{}'(0)=(2\alpha-1)B, H_1{}'(1^-)=2+\lim_{x\to 1}\left(abx\log\frac{e^{R+}}{1-x}-\log\frac{e^R}{1-x}\right)=\begin{cases}-\infty, & \text{if } ab<1\\ 2-c, & \text{if } ab=1\\ \infty, & \text{if } ab>1\end{cases} \tag{15}$$

a) Clearly, $f_1(0)=0$. By (10),

$$f_1(1^-)=\lim_{x\to 1}\left(x\log\frac{e^R}{1-x}-\lambda\log\frac{1}{1-x}\right)=\begin{cases}R, & \text{if } \lambda=1\\ \infty, & \text{if } \lambda<1\\ -\infty, & \text{if } \lambda>1\end{cases}.$$

b) In the case when $ab\leqslant\min\{1,c/2\}$, we see that $\alpha\leqslant 1/2$, and $c\geqslant 2$ if $ab=1$. Hence $e_n\geqslant 0$, and it follows from (12) that $H_1{}'$ is decreasing on $(0,1)$ with $H_1{}'(0)=(2\alpha-1)B\leqslant 0$, so that H_1 is decreasing from $(0,1)$ onto $(1-\lambda, B-\lambda)$. Therefore, by (11), the first two assertions in part b) follow.

Suppose that $ab\leqslant\min\{1,c/2\}$. Then by Lemma 1(a) and (11), all the coefficients of the Maclaurin series of f_1 are positive (negative) if $\lambda\leqslant 1$ ($\lambda\geqslant B$, respectively). This yields the absolute monotonicity properties of f_1.

c) Observe that the condition $ab\geqslant\max\{1,c/2\}$ implies that $\alpha\geqslant 1/2$, and $c\leqslant 2$ if $ab=1$. Hence it follows from (12) that $H_1{}'$ is strictly increasing on $(0,1)$ with $H_1{}'(0)=(2\alpha-1)B\geqslant 0$, so that H_1 is strictly increasing from $(0,1)$ onto $(B-\lambda, 1-\lambda)$. This shows that f_1 is increasing (decreasing) on $(0,1)$ if and only if $\lambda\leqslant B$ ($\lambda\geqslant 1$, respectively), and if $B<\lambda<1$, then there exists a number $r_2\in(0,1)$ such that f_1 is decreasing on $(0,r_2]$ and increasing on $[r_2,1)$.

Next, in this case, it follows from Lemma 1 that for $n\in\mathbb{N}_0$, $Ba_n-\lambda>0$ if $\lambda\leqslant B$, and $Ba_n-\lambda<0$ if $\lambda\geqslant 1$. This, together with (11), yields the absolute monotonicity properties of f_1 and $-f_1$.

d) The limiting values of f_2 follow from (11).

Suppose that $ab\leqslant\min\{3-c,(c+1)/3\}$. From (10), (12) and (15), we obtain the limiting values $H_2(0)=-f_1(0)=0, H_2(1^-)=H_1(1^-)=1-\lambda$.

$$H_3(0)=2\alpha B-\lambda, H_3(1^-)=\begin{cases}abR+2-c, & \text{if } \lambda=ab\\ \infty, & \text{if } \lambda<ab\\ -\infty, & \text{if } \lambda>ab\end{cases} \tag{16}$$

It follows from Lemma 1(b), (14) and (16) that H_3 is strictly increasing and convex (decreasing and concave) on $(0,1)$ if $\lambda\leqslant ab$ ($\lambda\geqslant 2\alpha B$, respectively), with $H_3(0)=2\alpha B-\lambda$. Hence by (14), if $\lambda\leqslant\min\{ab,2\alpha B\}$ ($\lambda\geqslant 2\alpha B$), then H_2 is increasing (decreasing) from $(0,1)$ onto $(0,1-\lambda)$ ($(1-\lambda,0)$, respectively). This, together with (13), yield the monotonicity properties of f_2.

e) The proof of part e) is similar to that of part d). □

2.2 Proof of Theorem 2

a) Clearly, $g_1(0)=1$, and if $\alpha\leqslant 1/4$, then

$$g_1(1^-)=\frac{\pi}{2}\lim_{x\to 1}\frac{\sin(\pi(1-x)^{2\alpha}F(x)/2)}{\pi(1-x)^{2\alpha}F(x)/2}\cdot\frac{F(x)}{(1-x)^{(1-4\alpha)/2}}=\frac{\pi}{2}\lim_{x\to 1}\frac{F(x)}{(1-x)^{(1-4\alpha)/2}}=\infty.$$

Let h_1 be as in Lemma 1(c), h_2 as in Lemma 2, and set $u=\pi(1-x)^{2\alpha}F(x)/2$. By differentiation,

$$\frac{2(1-x)^{3/2}}{u\cos u}g_1{}'(x)=G_1(x)\equiv h_1(u)+2\alpha h_2(x)-4\alpha.$$

By Lemmas 1(c) and Lemma 2, we see that the function G_1 is strictly decreasing $(0,1)$ onto $(1-4\alpha,\infty)$. Hence the monotonicity property of g_1 follows. The second assertion in part a) is clear.

b) Clearly, $g_3(0)=1$, and

$$g_3(1^-)=\frac{\pi}{2}\lim_{x\to 1}\frac{F(x)}{(1-x)^{\beta-\alpha}}\cdot\frac{\sin(\pi(1-x)^{\alpha}F(x)/2)}{\pi(1-x)^{\alpha}F(x)/2}=\begin{cases}\infty, & \text{if } \beta\geqslant\alpha\\ 0, & \text{if } \beta\geqslant 0\end{cases}.$$

Put $t=\pi(1-x)^{\alpha}F(x)/2$. Then by differentiation,

$$(1-x)^{\beta+1}g_3{}'(x)/\sin t=G_2(x)\equiv\beta-\alpha G_3(x),$$

where $G_3(x)=[h_1(t)]^{-1}\cdot[1-h_2(x)]$. By Lemmas 1(c) and Lemma 2, we see that $G_3(x)$ is a product of two positive and strictly increasing functions on $(0,1)$, with $G_3(0)=0$ and $G_3(1^-)=1$. This shows that G_2 is strictly decreasing from $(0,1)$ onto $(\beta-\alpha,\beta)$, and hence the conclusions in part b) follow.

c) Let $G_4(x)=\cos(\pi(1-x)^{2\alpha}F(x)/2)$ and $G_5(x)=x$. Then $G_4(0)=G_5(0)=0$, $g_4(x)=G_4(x)/G_5(x)$, and by differentiation,

$$\frac{2}{\pi\alpha}\cdot\frac{G_4{}'(x)}{G_5{}'(x)}=G_6(x)\equiv F(x)\cdot[2-h_2(x)]\cdot\frac{\sin(\pi(1-x)^{2\alpha}F(x)/2)}{(1-x)^{1-2\alpha}}.$$

By part (b) and Lemma 2(a), if $1-2\alpha\geqslant\alpha$, namely, $\alpha\leqslant 1/3$, then G_6 is a product of three positive and strictly increasing functions on $(0,1)$, so that G_6 is strictly increasing on $(0,1)$. Hence the monotonicity property of g_4 follows from [3, Theorem 1.25].

The remaining conclusions in part c) are clear. □

References:

[1] Owen D B, Abramowitz M, Stegun I A. Handbook of mathematical functions with formulas, graphs, and mathematical tables[J]. Technometrics, 1965, 7(1): 78.

[2] Anderson G D, Vamanamurthy M K, Vuorinen M. Conformal Invariants, Quasiconformal Maps, and Special Function [M]. Heidelberg: Springer Berlin Heidelberg, 1992: 1-19.

[3] Anderson G D, Vuorinen M. Conformal Invariants, Inequalities, and Quasiconformal Maps[M]. John Wiley and Sons, 1997:48-73.

[4] Qiu S L, Vuorinen M. Handbook of Complex Analysis: Special Function in Geometric Function Theory[M]. Landon: Elsevier Besloten Vennootschap, 2005: 621-659.

[5] Qiu S L, Vuorinen M. Infinite products and normalized quotients of hypergeometric functions[J]. SIAM Journal on Mathematical Analysis, 1999, 30(5): 1057-1075.

[6] Anderson G D, Vamanamurthy M K, Vuorinen M. Functional inequalities for hypergeometric functions and complete elliptic integrals[J]. SIAM Journal on Mathematical Analysis, 1992, 23(2): 512-524.

[7] Qiu S L, Vuorinen M. Landen inequalities for hypergeometric functions[J]. Nagoya Mathematical Journal, 1999, 154: 31-56.

[8] Qiu S L, Vuorinen M. Duplication inequalities for the ratios of hypergeometric functions[J]. Forum Mathematicum, 1999, 12(1): 109-133.

[9] Anderson G D, Qiu S L, Vamanamurthy M K, Vuorinen M. Generalized elliptic integrals and modular equations[J]. Pacific Journal of Mathematics, 2000, 192(1): 1-37.

[10] Huang T R, Qiu S L, Ma X Y. Monotonicity properties and inequalities for the generalized elliptic integral of the first kind[J]. Journal of Mathematical Analysis and Applications, 2019, 469(1): 95-116.

[11] Qiu S L, Ma X Y, Huang T R. Some properties of the difference between the Ramanujan constant and beta function[J]. Journal of Mathematical Analysis and Applications, 2017, 446(1): 114-129.

[12] 周丽明, 裘松良, 王飞. 完全椭圆积分 $K(r)$ 之由三角函数给出的界及其应用[J]. 浙江理工大学学报, 2012, 29(5): 722-726.

[13] Alzer H, Qiu S L. Monotonicity theorems and inequalities for the complete elliptic integrals[J]. Journal of Computational and Applied Mathematics, 2004, 172(2): 289-312.

[14] Alzer H, Richards K. Inequalities for the ratio of complete elliptic integrals [J]. Proceedings of the American Mathematical Society, 2017, 145(4): 1661-1670.

[15] Ponnusamy S, Vuorinen M. Asymptotic expansions and inequalities for hypergeometric function [J]. Mathematika, 1997, 44(2): 278-301.

[16] Bernstein S. Sur les fonctions absolument monotones[J]. Acta Mathematica, 1929, 52: 1-66.

(责任编辑:康　锋)