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Monotonicity and convexity properties of the generalized (p,q)-elliptic integrals

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Abstract: In this paper, the authors obtain several monotonicity and convexity properties of the generalized (p,q)-elliptic integrals $\mathscr{K}_{p,q}(r)$ and $\mathscr{E}_{p,q}(r)$ for $p,q\in(1,\infty)$ and $r\in(0,1)$, by studying the analytic properties of certain combinations in terms of $\mathscr{K}_{p,q}(r)$, $\mathscr{E}_{p,q}(r)$ and some elementary functions.

Key words: generalized (p,q)-trigonometric functions; generalized (p,q)-elliptic integrals; monotonicity; convexity

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0 Introduction

For $r\in(0,1)$, Legendre's complete elliptic integrals of the first and second kinds are defined as

$$\mathscr{K}=\mathscr{K}(r)=\int_0^{\pi/2}\frac{\mathrm{d}t}{\sqrt{1-r^2\sin^2 t}} \text{ and } \mathscr{E}=\mathscr{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2 t}\,\mathrm{d}t,$$

respectively, which are the special cases of the Gaussian hypergeometric functions

$$F(a,b;c;x)={}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!},\ |x|<1 \tag{1}$$

where $(a,0)=1$ for $a\neq0$, and (a,n) is the shifted factorial function

$$(a,n)=a(a+1)(a+2)\cdots(a+n-1) \tag{2}$$

for $n\in\mathbf{N}=\{k\mid k \text{ is a positive integer}\}$. (See [1])

In recent years, certain generalizations of the classical trigonometric functions have attracted much interest. For $p,q\in(1,\infty)$ and for $x\in[0,1]$, define the function

$$\operatorname{arcsin}_{p,q}x=\int_0^x\frac{\mathrm{d}t}{(1-t^q)^{1/p}} \tag{3}$$

and set

$$\pi_{p,q}=2\operatorname{arcsin}_{p,q}(1)=2\int_0^1\frac{\mathrm{d}t}{(1-t^q)^{1/p}}=\frac{2}{q}B\left(1-\frac{1}{p},\frac{1}{q}\right) \tag{4}$$

where B is the classical beta function. The function $\operatorname{arcsin}_{p,q}x$ has an inverse defined on $[0,\pi_{p,q}/2]$, which can be extended to an odd $2\pi_{p,q}$-periodic function, denoted by $\sin_{p,q}$, on the set $\mathbf{R}$ of real numbers by natural procedures designed to mimic the behaviour of the sine function. The function $\sin_{p,q}$ is said to be

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the generalized (p,q)-sine function, reduces to the classical sine function when $p=q=2$, and occurs as an eigenfunction of the Dirichlet problem for the (p,q)-Laplacian. (Cf. [2-3].)

For $p,q\in(1,\infty)$ and for $r\in(0,1)$, the so-called generalized (p,q)-elliptic integrals of the first and second kinds are defined as

$$\mathcal{K}_{p,q}(r)=\int_0^{\pi_{p,q}/2}\frac{\mathrm{d}t}{(1-r^q\sin_{p,q}^q t)^{1-1/p}}=\int_0^1\frac{\mathrm{d}t}{(1-t^q)^{1/p}(1-r^qt^q)^{1-1/p}} \tag{5}$$

and

$$\mathcal{E}_{p,q}(r)=\int_0^{\pi_{p,q}/2}(1-r^q\sin_{p,q}^q t)^{1/p}\mathrm{d}t=\int_0^1\left(\frac{1-r^qt^q}{1-t^q}\right)^{1/p}\mathrm{d}t \tag{6}$$

respectively, which were introduced and studied recently. (Cf. [4-5].) For $p=q=2$, these two functions reduce to the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$, respectively. It is easy to see that the generalized (p,q)-elliptic integrals have the following expressions (cf. [5]).

$$\begin{cases}\mathcal{K}_{p,q}=\mathcal{K}_{p,q}(r)=\dfrac{\pi_{p,q}}{2}F\left(1-\dfrac{1}{p},\dfrac{1}{q};1-\dfrac{1}{p}+\dfrac{1}{q};r^q\right)\\ \mathcal{K}'_{p,q}=\mathcal{K}'_{p,q}(r)=\mathcal{K}_{p,q}(r')\\ \mathcal{K}_{p,q}(0)=\dfrac{\pi_{p,q}}{2},\mathcal{K}_{p,q}(1)=\infty\end{cases} \tag{7}$$

and

$$\begin{cases}\mathcal{E}_{p,q}=\mathcal{E}_{p,q}(r)=\dfrac{\pi_{p,q}}{2}F\left(-\dfrac{1}{p},\dfrac{1}{q};1-\dfrac{1}{p}+\dfrac{1}{q};r^q\right)\\ \mathcal{E}'_{p,q}=\mathcal{E}'_{p,q}(r)=\mathcal{E}_{p,q}(r')\\ \mathcal{E}_{p,q}(0)=\dfrac{\pi_{p,q}}{2},\mathcal{E}_{p,q}(1)=1\end{cases} \tag{8}$$

Here and hereafter, we let $r'=(1-r^q)^{1/q}$. Clearly, $\mathcal{K}_{p,q}$ ($\mathcal{E}_{p,q}$) is strictly increasing (decreasing, respectively) on (0, 1). By [4, Corollary 1.2], these two functions satisfy the Legendre relation

$$\mathcal{K}_{p,q}(r)\mathcal{E}'_{p,q}(r)+\mathcal{K}'_{p,q}(r)\mathcal{E}_{p,q}(r)-\mathcal{K}_{p,q}(r)\mathcal{K}'_{p,q}(r)=\frac{\pi_{p,q}}{2}.$$

It is well known that the complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$ have many applications in several fields of mathematics as well as in physics and engineering. Numerous properties have been obtained for $\mathcal{K}$ and $\mathcal{E}$ (cf., for instance, [6-10]). However, only a few basic properties of the generalized (p,q)-elliptic integrals $\mathcal{K}_{p,q}$ and $\mathcal{E}_{p,q}$ have been revealed (see [4-5, 11]). It is natural to ask whether the known properties of $\mathcal{K}$ and $\mathcal{E}$ can been extended to $\mathcal{K}_{p,q}$ and $\mathcal{E}_{p,q}$.

The purpose of this paper is to present several monotonicity and convexity properties of $\mathcal{K}_{p,q}$ and $\mathcal{E}_{p,q}$, by studying the analytic properties of certain combinations defined in terms of $\mathcal{K}_{p,q}$, $\mathcal{E}_{p,q}$ and some elementary functions, thus extending some known properties of $\mathcal{K}$ and $\mathcal{E}$ to $\mathcal{K}_{p,q}$ and $\mathcal{E}_{p,q}$.

Throughout this paper, we always let $a=1-1/p$ and $b=a+1/q$ for $p,q\in(1,\infty)$, $\gamma=0.577215\cdots$ be the Euler constant, ψ the classical psi function, and let

$$R(x,y)=-2\gamma-\psi(x)-\psi(y)\,(x,y\in(0,\infty)) \tag{9}$$

1 Main Results

In this section, we state the main results of this paper.

Theorem 1 For $p,q\in(1,\infty)$, we have the following conclusions:

a) The function $f_1(r)\equiv(\mathcal{E}_{p,q}-r'^q\mathcal{K}_{p,q})/r^q$ is strictly increasing and convex from (0, 1) onto $(a\pi_{p,q}/(2b),1)$.

b) The function $f_2(r)\equiv r'^q\mathcal{K}_{p,q}/\mathcal{E}_{p,q}$ is strictly decreasing from (0, 1) onto itself.

c) The function $f_3(r)\equiv(\mathscr{E}_{p,q}-r'^q\mathscr{K}_{p,q})/(r^q\mathscr{K}_{p,q})$ is strictly decreasing from $(0,1)$ onto $(0,a/b)$.

d) The function $f_4(r)\equiv(\mathscr{K}_{p,q}-\mathscr{E}_{p,q})/(r^q\mathscr{K}_{p,q})$ is strictly increasing from $(0,1)$ onto $(1/(qb),1)$.

e) The function $f_5(r)\equiv(\mathscr{E}_{p,q}-r'^q\mathscr{K}_{p,q})/(\mathscr{K}_{p,q}-\mathscr{E}_{p,q})$ is strictly decreasing from $(0,1)$ onto $(0,aq)$.

f) The function $f_6(r)\equiv r'^q(\mathscr{K}_{p,q}-\mathscr{E}_{p,q})/(r^q\mathscr{E}_{p,q})$ is strictly decreasing from $(0,1)$ onto $(0,1/(qb))$.

Theorem 2 For $p,q\in(1,\infty)$ and $c\in(-\infty,\infty)$, we have the following conclusions:

a) The function $g_1(r)\equiv r'^c\mathscr{K}_{p,q}$ is decreasing (increasing) on $(0,1)$ if and only if $c\geqslant a/b$ ($c\leqslant 0$, respectively) with $g_1((0,1))=(0,\pi_{p,q}/2)$ if $c\geqslant a/b$.

b) The function $g_2(r)\equiv r'^c\mathscr{E}_{p,q}$ is increasing (decreasing) on $(0,1)$ if and only if $c\leqslant -1/(pb)$ ($c\geqslant 0$, respectively), with $g_2(0,1)=(\pi_{p,q}/2,\infty)$ if $c\leqslant -1/(pb)$.

Theorem 3 For $p,q\in(1,\infty)$ and $c\in(-\infty,\infty)$, the function $h_1(r)\equiv\mathscr{K}_{p,q}+c\log r'$ is increasing and convex (decreasing) on $[0,1)$ if and only if $c\leqslant a\pi_{p,q}/(2b)$ ($c\geqslant 1$, respectively). Moreover, if $c\geqslant 1$, then h_1 is concave on $[0,1)$. In particular, the function $h_2(r)\equiv\mathscr{K}_{p,q}+\log r'$ is strictly decreasing and concave from $(0,1)$ onto $(R(a,1/q)/q,\pi_{p,q}/2)$, so that for $p,q\in(1,\infty)$ and $r\in(0,1)$ and $R=R(a,1/q)/q$,

$$\frac{\pi_{p,q}}{2}+\log\frac{1}{r'}-\left(\frac{\pi_{p,q}}{2}-R\right)r\leqslant\mathscr{K}_{p,q}(r)\leqslant\frac{\pi_{p,q}}{2}+\log\frac{1}{r'} \tag{10}$$

with equality in each instance if and only if $r=0$.

Remark a) If $p=q=2$, then Theorem 1 a) and e), and Theorem 1 c)—d) reduce to [6, Theorem 3.21(1) & (6)] and [6, Exercise 3.43 (46) & (32)], respectively, while Theorems 2—3 reduce to [6, Theorem 3.21(7) & (8)] and [6, Theorem 3.21(3)], respectively.

b) If $p=q$ and $a=1/p$, then Theorems 1—3 give several properties of the generalized elliptic integrals $\mathscr{K}_a$ and $\mathscr{E}_a$, which were obtained in [12, Lemmas 5.2 & 5.4, Theorem 5.5(1)].

We now recall the following two lemmas needed in the proofs of our main results.

Lemma 1 ([13]) Let $r_n,s_n\in(-\infty,\infty)$ for $n\in\mathbf{N}$. Suppose that the power series

$$R(x)=\sum_{n=1}^{\infty}r_nx^n \text{ and } S(x)=\sum_{n=1}^{\infty}s_nx^n \tag{11}$$

are both convergent for $|x|<1$. If all $s_n>0$ and if r_n/s_n is strictly increasing (decreasing) in $n\in\mathbf{N}$, then the function R/S is strictly increasing (decreasing, respectively) on $(0,1)$.

Lemma 2 ([5]) For $p,q\in(1,\infty)$ and $r\in(0,1)$,

$$\frac{\mathrm{d}\mathscr{K}_{p,q}}{\mathrm{d}r}=\frac{\mathscr{E}_{p,q}-r'^q\mathscr{K}_{p,q}}{rr'^q},\ \frac{\mathrm{d}\mathscr{E}_{p,q}}{\mathrm{d}r}=-q\,\frac{\mathscr{K}_{p,q}-\mathscr{E}_{p,q}}{pr} \tag{12}$$

2 Proofs of Main Results

2.1 Proof of Theorem 1

a) By (1) and (7)—(8), we have

$$\begin{aligned}\mathscr{E}_{p,q}-r'^q\mathscr{K}_{p,q}&=\frac{\pi_{p,q}}{2}\left\{\sum_{n=0}^{\infty}\left[\frac{(-1/p,n)(1/q,n)}{(b,n)n!}-\frac{(a,n)(1/q,n)}{(b,n)n!}\right]r^{qn}+\sum_{n=0}^{\infty}\frac{(a,n)(1/q,n)}{(b,n)n!}r^{q(n+1)}\right\}\\&=\frac{\pi_{p,q}}{2}\sum_{n=1}^{\infty}\left[\frac{(-1/p,n)(1/q,n)}{(b,n)n!}-\frac{(a,n)(1/q,n)}{(b,n)n!}+\frac{(a,n-1)(1/q,n-1)}{(b,n-1)(n-1)!}\right]r^{qn}\\&=\frac{a\pi_{p,q}}{2}\sum_{n=1}^{\infty}\frac{n(a,n-1)(1/q,n-1)}{(b,n)n!}r^{qn}=\frac{a\pi_{p,q}}{2}r^q\sum_{n=0}^{\infty}b_nr^{qn}\end{aligned} \tag{13}$$

where $b_n=a_n/(n+b)$ and $a_n=(a,n)(1/q,n)/[(b,n)n!]$, and hence the monotonicity of f_1 follows. Clearly, $f_1(1^-)=1$. By (13), $f_1(0^+)=a\pi_{p,q}/(2b)$.

b) Observe that $\mathscr{E}_{p,q}-r'^q\mathscr{K}_{p,q}$ is increasing on $(0,1)$ by (13), and $\mathscr{E}_{p,q}$ is decreasing on $(0,1)$. Since $f_2(r)=1-(\mathscr{E}_{p,q}-r'^q\mathscr{K}_{p,q})/_{p,q}$, the monotonicity of f_2 follows. The limiting values of f_2 are clear.

c) It follows from (1), (7) and (13) that

$$f_3(r) = a\frac{\left(\sum_{n=0}^{\infty} b_n r^{qn}\right)}{\left(\sum_{n=0}^{\infty} a_n r^{qn}\right)} \tag{14}$$

Since $b_n/a_n = 1/(n+b)$ is strictly decreasing in $n \in \mathbf{N} \cup \{0\}$, we obtain the monotonicity of f_3 by Lemma 1.

The limiting values of f_3 are clear.

d) Since $f_4(r) = 1 - f_3(r)$, part d) follows from part c).

e) Part e) follows from parts c)—d).

f) Clearly, $f_6(r) = 1 - f_1(r)/\mathscr{E}_{p,q}$, and hence part f) follows from part a). □

2.2 Proof of Theorem 2

a) Let f_3 be as in Theorem 1 c). Then by Lemma 2 and by differentiation,

$$r'^{q-c} r^{1-q} g'_1(r)/\mathscr{K}_{p,q} = f_3(r) - c.$$

Hence by Theorem 1 c), g_1 is decreasing (increasing) on (0, 1) if and only if

$$c \geqslant \sup_{0<r<1} f_3(r) = \frac{a}{b}\ (c \leqslant \inf_{0<r<1} f_3(r) = 0, \text{respectively}).$$

b) Let f_6 be as in Theorem 1 f). Then by Lemma 2, we have

$$r'^{q-c} r^{1-q} g'_2(r)/\mathscr{E}_{p,q} = -[c + q f_6(r)/p].$$

Hence by Theorem 1 f), g_2 is increasing (decreasing) on (0, 1) if and only if

$$c \leqslant -\frac{q}{p}\sup_{0<r<1} f_6(r) = -\frac{1}{bp}\left(c \geqslant -\frac{q}{p}\inf_{0<r<1} f_6(r) = 0, \text{respectively}\right),$$

yielding the result in part b) as desired. □

2.3 Proof of Theorem 3

Let f_1 be as in Theorem 1 a). Then Lemma 2 and differentiation give

$$r'^{q} r^{1-q} h'_1(r) = f_1(r) - c,$$

so that by Theorem 1 a),

$$h'_1(r) \geqslant 0\ (\leqslant 0) \Leftrightarrow c \leqslant \inf_{0<r<1} f_1(r) = \frac{a\pi_{p,q}}{2b}\ (c \geqslant \sup_{0<r<1} f_1(r) = 1, \text{respectively}),$$

which yields the monotonicity of h_1.

Next, it follows from (1) and (7) that

$$h_1(r) - \frac{\pi_{p,q}}{2} = \frac{1}{q}\sum_{n=1}^{\infty}\frac{1}{n}(A_n - c) r^{qn} \tag{15}$$

where $A_n = q\pi_{p,q}(a,n)(1/q,n)/[2(b,n)(n-1)!]$. Since

$$\frac{A_{n+1}}{A_n} = 1 + \frac{a}{qn(n+b)} > 1,$$

A_n is strictly increasing in $n \in \mathbf{N}$. Clearly, $A_1 = q\pi_{p,q}/(2b)$. By [1, 6.1.47] and (4), it is easy to obtain the limiting value $A_\infty = \lim_{n\to\infty} A_n = 1$. Hence it follows from (15) that h_1 is convex on (0, 1) if and only if $c \leqslant A_1 = q\pi_{p,q}/(2b)$, and h_1 is concave on (0, 1) if $c \geqslant A_\infty = 1$.

The limiting value $h_2(0) = \pi_{p,q}/2$, and the monotonicity and concavity properties of h_2 are clear. By [1, 15.3.10], we obtain the limiting value $h_2(1^-) = R(a, 1/q)/q$.

The double inequality and its equality case are clear. □

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广义(p,q)-椭圆积分的单调性和凹凸性

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摘要：通过研究由广义(p,q)-三角函数定义的一种新型的广义(p,q)-椭圆积分$\mathscr{K}_{p,q}$、$\mathscr{E}_{p,q}$与某些初等函数的组合的分析性质，获得了$\mathscr{K}_{p,q}$和$\mathscr{E}_{p,q}$的一些单调性和凹凸性。其中，$p,q\in(1,\infty)$，$r\in(0,1)$。

关键词：广义(p,q)-三角函数；广义(p,q)-椭圆积分；单调性；凹凸性

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