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A double inequality for the modulus of the Grötzsch ring in $\mathbf{R}^n$

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Abstract: Let $r'=\sqrt{1-r^2}$ and $M_n(r)$ be the (conformal) modulus of the Grötzsch Ring in the quasiconformal theory in $\mathbf{R}^n$, for $n\geqslant 3$ and $r\in(0,1)$. In this paper, a double inequality is obtained for the function $H(r)\equiv r'^2M_n(r)M_n(r')^{n-1}+r^2M_n(r')M_n(r)^{n-1}$, thus improving known bounds for $H(r)$, and correcting an error in the proof of a related inequality for $H(r)$ which was given in a monograph by G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen.

Key words: n-dimensional quasiconformal theory; the Grötzsch ring; modulus; inequalities

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0 Notation and Main Results

For $n\geqslant 2$, let $\mathbf{R}^n$ denote the n-dimensional Euclidian space, $\overline{\mathbf{R}}^n=\mathbf{R}^n\cup\{\infty\}$, B^n the unit ball in $\mathbf{R}^n$, and let $e_1, e_2, \ldots, e_n$ be the standard unit vectors in $\mathbf{R}^n$. A domain $D\subset\overline{\mathbf{R}}^n$ is said to be a ring domain(or a ring in brief) if $\overline{\mathbf{R}}^n\setminus D$ consists of two components C_0 and C_1, where C_0 is bounded. Such a ring is usually denoted by $R(C_0,C_1)$. For $s>1$, the so-called Grötzsch ring is defined by

$$R_{G,n}(s)=R(\overline{B}^n,[se_1,\infty]),s>1,$$

which means that the complementary components of the Grötzsch ring $R_{G,n}(s)$ with respect to $\overline{\mathbf{R}}^n$ are $C_0=\overline{B}^n=B^n\cup\partial B^n$ and $C_1=[se_1,\infty]$. (See [1, p. 149].)

For $E,F\subset G\subset\overline{\mathbf{R}}^n$, we denote the family of curves joining E and F in G by $\Delta(E,F;G)$. If $G=\mathbf{R}^n$ or $\overline{\mathbf{R}}^n$, then we may omit G and simply denote $\Delta(E,F;G)$ by $\Delta(E,F)$. Let Γ be a family of curves in $\overline{\mathbf{R}}^n$, $\overline{\mathbf{R}}=\mathbf{R}\cup\{\infty\}$, and for an arbitrary locally rectifiable curve $\gamma\in\Gamma$, put $\mathscr{F}(\Gamma)=\{\rho\,|\,\rho:\mathbf{R}^n\to\overline{\mathbf{R}}$ is a nonnegative Borel-measurable function such that $\int_\gamma\rho\mathrm{d}s\geqslant 1\}$. The function ρ is said to be admissible if $\rho\in\mathscr{F}(\Gamma)$. The modulus of Γ is then defined as

$$M(\Gamma)=\inf_{\rho\in\mathscr{F}(\Gamma)}\int_{\mathbf{R}^n}\rho^n\mathrm{d}m,$$

where m is the n-dimensional Lebesgue measure. By [1, Theorem 8.28, (8.31), (8.34) and (8.35)], the conformal capacity cap $R_{G,n}(s)$ of the Grötzsch ring $R_{G,n}(s)$ can be expressed by

$$\gamma_n(s)\equiv\mathrm{cap}\,R_{G,n}(s)\equiv M(\Delta(B^n,[se_1,\infty])),$$

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while the (conformal) modulus of $R_{G,n}(1/r)$ is defined by

$$M_n(r)=\operatorname{mod} R_{G,n}(1/r)=\left[\frac{\omega_{n-1}}{\gamma_n(1/r)}\right]^{1/(n-1)}, r\in(0,1),$$

where ω_{n-1} is the surface area of the unit sphere $S^{n-1}=\partial \mathrm{B}^n$. Clearly, $\mu(r)\equiv M_2(r)$ is exactly the so-called Grötzsch ring function, which has the following expression

$$\mu(r)=\frac{\pi}{2}\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}, \tag{1}$$

where

$$\mathscr{K}(r)=\int_0^{\pi/2}\frac{\mathrm{d}t}{\sqrt{1-r^2\sin^2 t}} \text{ and } \mathscr{K}'(r)=\mathscr{K}(r')$$

for $r\in(0,1)$, are the complete elliptic integrals of the first kind (see [1] or [2]). Here and here-after, we always let $r'=\sqrt{1-r^2}$ for $r\in[0,1]$. It is well known that the Grötzsch ring $R_{G,n}(1/r)$ and its modulus $M_n(r)$ or its capacity $\gamma_n(1/r)$ play an extremely important role in the study of quasiconformal mappings in $\mathbf{R}^n$.

The Grötzsch ring constant λ_n is defined by

$$\log\lambda_n=\lim_{r\to 0^+}[M_n(r)+\log r],$$

which is indispensable in the study of $M_n(r)$ and $\gamma_n(s)$. It is well known that $\lambda_2=4$. Unfortunately, so far we have only known some estimates for λ_n when $n\geqslant 3$, among which is the following double inequality

$$2\mathrm{e}^{0.76(n-1)}<\lambda_n\leqslant 2\mathrm{e}^{n+(1/n)-(3/2)}, n\geqslant 3 \tag{2}$$

(see [1, Theorem 12.21(1)] and [3]).

Now we introduce the gamma and beta functions, and some constants depending only on n, which are needed in the study of the properties of $M_n(r)$ and $\gamma_n(s)$. As usual, for complex numbers x and y with Re $x>0$ and Re $y>0$, the gamma and beta functions are defined by

$$\Gamma(x)=\int_0^{\infty}t^{x-1}\mathrm{e}^{-t}\mathrm{d}t \text{ and } B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t,$$

respectively. (Cf. [4] and [5].) It is well known that, for $n\geqslant 3$, the volume Ω_n of B^n and the $(n-1)$-dimensional surface area ω_{n-1} of S^{n-1} can be expressed by

$$\Omega_n=\frac{2\pi}{n}\Omega_{n-2}=\frac{\pi^{n/2}}{\Gamma(1+n/2)} \text{ and } \omega_{n-1}=n\,\Omega_n=\frac{n\,\pi^{n/2}}{\Gamma(1+n/2)},$$

respectively. (Cf. [1, 2.23] and [6].) Let

$$J_n=\int_0^{\pi/2}(\sin t)^{(2-n)/(n-1)}\mathrm{d}t=\frac{1}{2}B\left(\frac{1}{2(n-1)},\frac{1}{2}\right), c_n=(2J_n)^{1-n}\omega_{n-2}, A_n=\left(\frac{\omega_{n-1}}{2^n c_n}\right)^{1/(n-1)}.$$

In particular,

$$J_2=\pi/2, J_3=\sqrt{2}\mathscr{K}(1/\sqrt{2})=2.62205\cdots, c_2=2/\pi, c_3=4\pi^2\Gamma(1/4)^{-4}=0.22847\cdots, A_2=\pi^2/4 \text{ and } A_3=J_3.$$

Some properties of Ω_n, ω_{n-1}, J_n, c_n and A_n were given in [1, pp. 38-44 & 163] and in [6].

In the sequel, we let arth denote the inverse function of the hyperbolic tangent tanh, that is,

$$\operatorname{arth} x=\frac{1}{2}\log\frac{1+x}{1-x}, \ -1<x<1.$$

During the past decades, many properties have been obtained for $\mu(r)$ (cf. [1]-[2] and [7]). The known properties of $M_n(r)$, however, are much less than those of $\mu(r)$, because of lack of effective tools for the study of $M_n(r)$ when $n\geqslant 3$. For example, we have no explicit expression as or similar to (1) for $M_n(r)$ when $n\geqslant 3$. For the known properties of $M_n(r)$ and its related functions, the reader is referred to [1], [3] and [7-13]. Some of these known results for $M_n(r)$ are related to the constants λ_n, Ω_n, ω_{n-1}, J_n, c_n and A_n. For example, the following inequalities hold

$$A_n\left[\frac{1}{2}\mu\left(\frac{1-r}{1+r}\right)\right]^{1/(1-n)}\leqslant M_n(r)\leqslant A_n\left(\frac{1}{2}\log\frac{1-r}{1+r}\right)^{1/(1-n)} \tag{3}$$

$$\log\frac{1+r'}{r}<M_n(r)<\log\frac{\lambda_n(1+r')}{2r} \tag{4}$$

$$0<M_n(r)^{n-1}\log\frac{1+r}{1-r}<2A_n^{n-1} \tag{5}$$

for $r\in(0,1)$ and $n\geqslant 3$ (see [1, Theorems 11.20(1), 11.21(2)&(4), and 11.21(5)]).

On the other hand, if we let $h_n(r)=r'^2M_n(r)M_n(r')^{n-1}$, then for all $r\in(0,1)$,

$$h_2(r)+h_2(r')=\mu(r)\mu(r')\equiv\pi^2/4$$

by [1, (5.2)]. It is well known that for each $n\geqslant 2$, all $r\in(0,1)$ and for all $K>0$,

$$\varphi_{K,n}(r)^2+\varphi_{1/K,n}(r')^2=1\Leftrightarrow M_n(r)M_n(r')=\text{const},$$

where $\varphi_{K,n}(r)=M_n^{-1}(\alpha M_n(r))$ and $\alpha=K^{1/(1-n)}$ (cf. [1, 8.70]). Therefore, it is quite significant for us to study the properties of the function h_n, in order to reveal the properties of $M_n(r)$ and $\varphi_{K,n}(r)$. In [8, Theorem 5.1(3)], it was proved that for each $n\geqslant 2$ and all $r\in(0,1)$,

$$A_n^{n-1}=\frac{\omega_{n-1}}{2^nc_n}<h_n(r)+h_n(r')<\frac{4\omega_{n-1}}{2^nc_n}\log\lambda_n=4A_n^{n-1}\log\lambda_n \tag{6}$$

Later, [1, 11.36(2)] says that for each $n\geqslant 2$ and all $r\in(0,1)$,

$$A_n^{n-1}=\frac{\omega_{n-1}}{(2^nc_n)}<h_n(r)+h_n(r')<2A_n^{n-1}\log\lambda_n \tag{7}$$

However, the proof of the second inequality in (7) given in [1, p. 244] contains an error. This proof in [1, p. 244] is as follows: [1, Corollary 11.23(1) and (4)] yield

$$h_n(r)\leqslant A_n^{n-1}r'^2\,\frac{\log(\lambda_n/r)}{\log(1/r)},$$

and the upper bound in (7) follows, since [1, Theorem 1.25] implies that the function

$$r\mapsto r'^2\,\frac{\log(\lambda_n/r)}{\log(1/r)}$$

is increasing from $(0,1)$ onto $(1,2\log\lambda_n)$. It is easy to see that by this "proof", one can only obtain the following inequality

$$h_n(r)<2A_n^{n-1}\log\lambda_n,$$

so that the upper bound for $h_n(r)+h_n(r')$, which we can obtain by this method, is as follows

$$h_n(r)+h_n(r')<4A_n^{n-1}\log\lambda_n,$$

consisting with that in (6). So far, the known best upper bound for $h_n(r)+h_n(r')$ is given by (6).

In addition to indicating the error in the proof of (7) given in [1, p. 244] as above-mentioned, the main purpose of this paper is to improve the upper bound given in (6) by proving the following result.

Theorem 1 Let $h_n(r)=r'^2M_n(r)M_n(r')^{n-1}$. Then for each $n\geqslant 2$ and all $r\in(0,1)$,

$$A_n^{n-1}<h_n(r)+h_n(r')<\beta A_n^{n-1}\log\lambda_n \tag{8}$$

where

$$\beta=\frac{1}{\log(1+\sqrt{2})}\left[1+\frac{\log(1+\sqrt{2})-\log 2}{1.52+\log 2}\right]=1.23108\cdots.$$

1 Proof of Theorem 1

The proof of Theorem 1 stated in Section 0 requires the following lemma.

1.1 A Technical Lemma

Lemma 1 a) For $r\in(0,1)$, let $g(r)=r^2/\text{arth}\,r$ and $f(r)=g'(r)/r$. Then f is strictly decreasing from $(0,1)$ onto $(-\infty,\infty)$.

b) The function $F(r)\equiv g(r)+g(r')$ is strictly increasing on $\left(0,\frac{1}{\sqrt{2}}\right]$, and decreasing on $\left[\frac{1}{\sqrt{2}},1\right)$. In particular, for all $r\in(0,1)$,

$$F(r)\leqslant F\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{\log(1+\sqrt{2})} \tag{9}$$

The first equality in (9) holds if and only if $r=1/\sqrt{2}$.

Proof: a) Differentiation gives

$$g'(r)=\frac{r}{(\operatorname{arth} r)^2}\left(2\operatorname{arth} r-\frac{r}{r'^2}\right),$$

so that

$$f(r)=\frac{g'(r)}{r}=\frac{2}{\operatorname{arth} r}-\frac{r}{(r'\operatorname{arth} r)^2} \tag{10}$$

Clearly, $f(0^+)=\infty$ and $f(1^-)=-\infty$. By differentiation,

$$\frac{r'}{r}(r'\operatorname{arth} r)^3 f'(r)=2\left(1-\frac{\operatorname{arth} r}{r}\right)-\frac{r'^2\operatorname{arth} r}{r} \tag{11}$$

which is negative for all $r\in(0,1)$ since the function $r\mapsto(\operatorname{arth} r)/r$ is strictly increasing from $(0,1)$ onto $(1,\infty)$. This yields the result for f.

b) It is easy to verify that

$$\frac{1}{r}F'(r)=h(r)\equiv f(r)-f(r').$$

By part (1), h is strictly decreasing from $(0,1)$ onto $(-\infty,\infty)$ and has a unique zero $r_0=1/\sqrt{2}$ on $(0,1)$. This yields the piecewise monotonicity of F.

Then the remaining conclusions are clear.

1.2 Proof of Theorem 1

The first inequality in (8) was proved in [8, Theorem 5.1(3)].

Let $H(r)=h_n(r)+h_n(r')$, and F be as in Lemma 1 b). By (5), we see that

$$M_n(r)^{n-1}\operatorname{arth} r<A_n^{n-1},\ n\geqslant 2,\ 0<r<1 \tag{12}$$

On the other hand, the following inequality holds

$$M_n(r)<\log(\lambda_n/2)+\operatorname{arth} r' \tag{13}$$

for each $n\geqslant 2$ and all $0<r<1$, since the function

$$r\mapsto M_n(r)/[\log(\lambda_n/2)+\operatorname{arth} r']$$

is strictly decreasing from $(0,1)$ onto $(0,1)$ by [1, Theorem 11.21(4)]. It follows from (12) and (13) that

$$\begin{aligned}
H(r)&=\frac{r'^2M_n(r)}{\operatorname{arth} r'}\cdot M_n(r')^{n-1}\operatorname{arth} r'+\frac{r^2M_n(r')}{\operatorname{arth} r}\cdot M_n(r)^{n-1}\operatorname{arth} r\\
&\leqslant A_n^{n-1}\left[\frac{r'^2}{\operatorname{arth} r'}M_n(r)+\frac{r^2}{\operatorname{arth} r}M_n(r')\right]\\
&\leqslant A_n^{n-1}\left[\frac{r'^2}{\operatorname{arth} r'}\left(\log\frac{\lambda_n}{2}+\operatorname{arth} r'\right)+\frac{r^2}{\operatorname{arth} r}\left(\log\frac{\lambda_n}{2}+\operatorname{arth} r\right)\right]\\
&=A_n^{n-1}\left[1+\left(\frac{r^2}{\operatorname{arth} r}+\frac{r'^2}{\operatorname{arth} r'}\right)\log\frac{\lambda_n}{2}\right]\\
&=A_n^{n-1}\left[1+F(r)\log\frac{\lambda_n}{2}\right].
\end{aligned}$$

This, together with Lemma 1 b), yields

$$H(r)\leqslant A_n^{n-1}\left[1+\frac{1}{\log(1+\sqrt{2})}\log\frac{\lambda_n}{2}\right]=A_n^{n-1}\,\frac{\log\lambda_n}{\log(1+\sqrt{2})}\left[1+\frac{\log(1+\sqrt{2})-\log 2}{\log\lambda_n}\right] \tag{14}$$

By (2), the following double inequality holds

$$\frac{1}{\log \lambda_n} < \frac{1}{0.76(n-1)+\log 2} \leqslant \frac{1}{1.52+\log 2} \tag{15}$$

with equality if and only if $n=3$. Since $\log(1+\sqrt{2})-\log 2=0.188226\cdots>0$, it follows from (14) and (15) that

$$H(r) \leqslant \beta A_n^{n-1} \log \lambda_n,$$

where

$$\beta=\frac{1}{\log(1+\sqrt{2})}\left[1+\frac{\log(1+\sqrt{2})-\log 2}{1.52+\log 2}\right]=1.23108\cdots.$$

This yields the second inequality in (8) as desired.

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$\mathbf{R}^n$ 中 Grötzsch 环的共形模的一个不等式

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摘要：设 $M_n(r)$ 为 n 维拟共形理论中的 Grötzsch 环 $R_{G,n}(1/r)$ 的模，$r'=\sqrt{1-r^2}$，其中 $0<r<1$，$n\geqslant 3$。建立了函数 $H(r)\equiv r'^2 M_n(r)M_n(r')^{n-1}+r^2 M_n(r')M_n(r)^{n-1}$ 满足的一个双向不等式，较大程度地改进了 $H(r)$ 的已知上界，指出并纠正了 G. D. Anderson、M. K. Vamanamurthy 和 M. Vuorinen 的专著中给出的关于 $H(r)$ 的一个上界的证明中存在的错误。

关键词：n 维拟共形理论；Grötzsch 环；共形模；不等式

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