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Estimates for Complete Elliptic Integral $\mathcal{K}(r)$ by Trigonometric Functions with Applications*

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Abstract: In this paper，several lower and upper bounds of the complete elliptic integral of the first kind $\mathcal{K}(r)$ are obtained in terms of trigonometric functions sine and cosines，as well as those of Hübner's upper bound function. New estimates for the Hersch-Puger distortion function $\varphi_K(r)$ are presented according to the experimental results.

Key words: estimates；complete elliptic integrals；Hersch-Puger distortion function；quasiconformal mappings

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0 Introduction

Throughout this paper，we let $r'=\sqrt{1-r^2}$ for $r\in[0,1]$. The complete elliptic integrals of the first and second kinds are defined by [1-2]

$$\begin{cases}\mathcal{K}=\mathcal{K}(r)=\int_0^{\frac{\pi}{2}}(1-r^2\sin^2\theta)^{-\frac{1}{2}}\,d\theta,\\ \mathcal{K}'=\mathcal{K}'(r)=\mathcal{K}(r'),\\ \mathcal{K}(0)=\pi/2,\mathcal{K}(1)=\infty\end{cases}\tag{1}$$

and

$$\begin{cases}\varepsilon=\varepsilon(r)=\int_0^{\frac{\pi}{2}}(1-r^2\sin^2\theta)^{\frac{1}{2}}\,d\theta,\\ \varepsilon'=\varepsilon'(r)=\varepsilon(r'),\\ \varepsilon(0)=\frac{\pi}{2},\varepsilon(1)=1,\end{cases}\tag{2}$$

respectively.

It is well known that the complete elliptic integrals have many important applications in physics，engineering，geometric function theory，quasiconformal analysis，theory of mean values，number theory and many other related fields. For these，and for properties of $\mathcal{K}(r)$ and $\varepsilon(r)$，see [3-10].

Let B^2 be the unit disk in the plane，and $\mu(r)$ be the modulus of the Grötzsch ring $B^2\backslash[0,r]$ for $r\in(0,1)$. Then [4]

$$\mu(r)=\frac{\pi\mathcal{K}'(r)}{2\mathcal{K}(r)}.\tag{3}$$

For $K\in(0,\infty)$，the Hersch-Pfluger distortion function φ_K on $[0,1]$ is defined by [4]

$$\varphi_K(r)=\mu^{-1}\left(\frac{\mu(r)}{K}\right)\text{ for } r\in(0,1),$$
$$\varphi_K(0)=\varphi_K(1)-1=0.\tag{4}$$

It is well known that the Hersch-Pfluger distortion function $\varphi_K(r)$ plays an extremrly important role in quasiconformal theory as well as in some other mathematical fields. (Cf. [4].) This quasiconformal special function has been the subject of intensive researches，and many properties

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including estimates have been obtained. (See, for example, [4, 11-15]).

Let $m(r)=\left(\frac{2}{\pi}\right)r'^2\mathcal{K}(r)\mathcal{K}'(r)$ for $0<r<1$. O. Hübner obtained the following important inequality[11]

$$\varphi_K(r)<r^{\frac{1}{K}}\exp\left\{\left(1-\frac{1}{K}\right)[m(r)+\log(r)]\right\} \tag{5}$$

for all $r\in(0,1)$ and $K\in(1,\infty)$.

In [12], the following conclusions were proved: For all $r\in(0,1)$ and $K\in(1,\infty)$,

$$\varphi_K(r)<r^{\frac{1}{K}}\exp\left\{\left(1-\frac{1}{K}\right)a(r)\right\} \tag{6}$$

holds if and only if $a(r)\geqslant m(r)+\log r$, and

$$\varphi_{\frac{1}{K}}(r)<r^K\exp\{(1-K)b(r)\} \tag{7}$$

holds if and only if $b(r)\leqslant m(r)+\log r$, where both $a(r)$ and $b(r)$ and real functions on $(0,1)$.

The main purpose of this paper is to present several lower and upper bounds of the complete elliptic integral of the first kind $\mathcal{K}(r)$ in terms of trigonometric functions sine and cosines. We shall also obtain the bounds for the Hübner upper bound function $m(r)+\log r$, and apply these results to provide new extimates for the Hersch-Pfluger distortion function $\varphi_K(r)$.

1 Lemmas

In order to prove our main results, we need some formulas and lemmas, which are presented in this section.

First we state the following derivative formulas [4, pp. 474-475]

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}r}=\frac{\varepsilon-r'^2\mathcal{K}}{rr'^2},\frac{\mathrm{d}\varepsilon}{\mathrm{d}r}=\frac{\varepsilon-\mathcal{K}}{r},$$

$$\frac{\mathrm{d}(\varepsilon-r'^2\mathcal{K})}{\mathrm{d}r}=r\mathcal{K},\frac{\mathrm{d}(\mathcal{K}-\varepsilon)}{\mathrm{d}r}=\frac{r\varepsilon}{r'^2},$$

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r}=\frac{1}{r}\left(1-\frac{4}{\pi}\mathcal{K}\varepsilon'\right)=\frac{1}{r}\left[\frac{4\mathcal{K}'(\varepsilon-\mathcal{K})}{\pi}-1\right]$$

for $0<r<1$, and the following two lemmas, (see [4, Theorems 1.25, 3.21(7) & 3.30(2), Exercise 3.43(11)] and [14, Lemma]).

Lemma 1. For $-\infty<a<b<\infty$, let f, $g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b), with $g'(x)\neq 0$ on (a,b). If $\frac{f'(x)}{g'(x)}$ is increasing (decreasing) on (a,b), then so are

$$\frac{f(x)-f(a)}{g(x)-g(a)} \text{ and } \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $\frac{f'(x)}{g'(x)}$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.

(1) $\sqrt{r'}\mathcal{K}$ is strictly decreasing from $(0,1)$ onto $\left(0,\frac{\pi}{2}\right)$;

(2) $\frac{m(r)}{\log\left(\frac{1}{r}\right)}$ is trictly increasing from $(0,1)$ onto $(1,\infty)$;

(3) $\frac{(\mathcal{K}-\varepsilon)}{r^2}$ is strictly increasing from $(0,1)$ onto $\left(\frac{\pi}{4},\infty\right)$;

(4) $\frac{\left[r\log\left(\frac{1}{r}\right)\right]}{r'^2}$ is strictly increasing from $(0,1)$ onto $\left(0,\frac{1}{2}\right)$.

Next, we prove a technical lemma needed later.

Lemma 3.

(1) The function $f(r)\equiv\frac{\left[\varepsilon\log\left(\frac{4}{r'}\right)-\mathcal{K}\right]}{\left[r'\log\left(\frac{4}{r'}\right)\right]}$ is strictly decreasing from $(0,1)$ onto $\left(0,\frac{\pi\left[1-\left(\frac{1}{\log 4}\right)\right]}{2}\right)$.

(2) The function $g(r)\equiv\frac{f(r)}{r'}$ is strictly increasing from $(0,1)$ onto $\left(\frac{\pi\left[1-\left(\frac{1}{\log 4}\right)\right]}{2},\infty\right)$. In particular, for all $r\in(0,1)$,

$$\frac{\pi}{2}\left(1-\frac{1}{\log 4}\right)r'^2<\varepsilon-\frac{\mathcal{K}}{\log\left(\frac{4}{r'}\right)}<\frac{\pi}{2}\left(1-\frac{1}{\log 4}\right)r'. \tag{8}$$

(3) The function $h(r)\equiv\frac{(\mathcal{K}-\varepsilon)}{\left[\log\left(\frac{4}{r'}\right)-1\right]}$ is strictly increasing from $(0,1)$ onto $(0,1)$.

Proof. (1) By [4, (3.25)],

$$\lim_{r\to1}\left[\varepsilon log\left(\frac{4}{r'}\right)-\mathcal{K}\right]=$$

$$\lim_{r\to1}\left[\frac{\varepsilon-1}{r'}.\left(r'\log\frac{4}{r'}\right)-\left(\mathcal{K}-\log\frac{4}{r'}\right)\right]=0.$$

Let $f_1(r)=\varepsilon\log\left(\frac{4}{r'}\right)-\mathcal{K}$, $f_2(r)=r'\log\left(\frac{4}{r'}\right)$, $f_3(r)=\mathcal{K}-\varepsilon$ and $f_4(r)=\frac{r^2}{r'}$. Then $f_1(1^-)=f_2(1^-)=f_3(0)=f_4(0)=0$, $f(r)=\frac{f_1(r)}{f_2(r)}$ and

$$\frac{f'_1(r)}{f'_2(r)}=\frac{f_3(r)}{f_4(r)},\frac{f'_3(r)}{f'_4(r)}=\frac{r'\varepsilon}{2-r^2}. \qquad (9)$$

Clearly, the function $r\to\frac{r'}{(2-r^2)}$ is positive and strictly decreasing on (0,1), and hence so is f by Lemma 1 and (9).

It is clear that $f(0)=\pi\frac{\left[1-\left(\frac{1}{\log4}\right)\right]}{2}$. By l'Hôpital's Rule and (9), $f(1^-)=0$.

(2) Let $f_5(r)=r'^2\log\left(\frac{4}{r'}\right)$. Then $f_5(1^-)=f_1(1^-)=0$, $g(r)=\frac{f_1(r)}{f_5(r)}$ and

$$\frac{f'_1(r)}{f'_5(r)}=\frac{\mathcal{K}-\varepsilon}{r^2}.\frac{\log\left(\frac{4}{r'}\right)-1}{2\log\left(\frac{4}{r'}\right)-1}. \qquad (10)$$

It is easy to show that the function $r\to\frac{\left[\log\left(\frac{4}{r'}\right)-1\right]}{\left[2\log\left(\frac{4}{r'}\right)-1\right]}$ is strictly increasing on (0,1) so that the function on the right side of (10) is strictly increasing on (0,1) by [6, Lemma 5.2(3)]. Hence the monotonicity of g follows from Lemma 1.

Clearly, $g(0)=\pi\frac{\left[1-\left(\frac{1}{\log4}\right)\right]}{2}$. By l'Hôpital's Rule, $g(1^-)=\infty$.

(3) It is clear that $h(1^-)=1$. By l'Hôpital's Rule, $h(0)=0$. Diffrerntiation gives

$$h'(r)=\frac{r}{r'^2\left[\log\left(\frac{4}{r'}\right)-1\right]^2}f_1(r), \qquad (11)$$

and hence the monotoicity of h follows from part (1).

2 Main Results

Theorem 1. The function $F(r)\equiv\frac{[m(r)+\log r]}{[r'\mathcal{K}(r)]}$ is strictly decreasing and concave from (0,1) onto $\left(0,\frac{(\log16)}{\pi}\right)$. In particular, for $r\in(0,1)$,

$$\frac{\log16}{\pi}(1-r)r'\mathcal{K}(r)<m(r)+\log r<\frac{\log16}{\pi}r'\mathcal{K}(r). \qquad (12)$$

Proff. By differentiation,

$$F'(r)=-\frac{\mathcal{K}-\varepsilon}{r^2}\cdot\frac{r}{r'^2}\log\frac{1}{r}\cdot\frac{1}{r'\mathcal{K}^2}\cdot\left[\frac{m(r)}{\log\left(\frac{1}{r}\right)}+1\right], \qquad (13)$$

which is strictly decreasing from (0,1) onto $(-\infty,0)$ by Lemma 2. Therefore, F is strictly decreasing and concave on (0,1).

Clearly, $F(0^+)=\frac{(\log16)}{\pi}$. By l'Hôpital's Rule, $F(1^-)=0$. the double inequality (12) is obvious.

Theorem 2. (1) The functions $G_1(r)\equiv\frac{[\sin(r'\mathcal{K})]}{r'}$ and $G_2(r)\equiv\frac{[\cos(r'\mathcal{K})]}{r^2}$ are both strictly increasing on (0,1), with ranges $(1,\infty)$ and $\left(\frac{\pi}{8},1\right)$, respectively.

(2) The function $H(r)\equiv\frac{[\sin(r'\mathcal{K})]}{\left[r'\log\left(\frac{4}{r'}\right)\right]}$ is strictly increasing from (0,1) onto $\left(\frac{1}{\log 4},1\right)$. In particular, for $r\in(0,1)$,

$$\max\left\{\frac{\arccos(r^2)}{r'},\frac{\arcsin[P(r)]}{r'}\right\}<\mathcal{K}(r)<\frac{\arccos\left(\frac{\pi r^2}{8}\right)}{r'}, \qquad (14)$$

where $P(r)=\frac{\left[r'\log\left(\frac{4}{r'}\right)\right]}{\log4}$.

Proof. (1) Let $G_3(r)=\sin(r'\mathcal{K})$ and $G_4(r)=r'$. Then $G_1(r)=\frac{G_3(r)}{G_4(r)}$, $G_3(1^-)=G_4(1)=0$, and

$$\frac{G'_3(r)}{G'_4(r)}=\frac{\mathcal{K}-\varepsilon}{r^2}\cos(r'\mathcal{K}), \qquad (15)$$

which is strictly increasing on (0,1) by Lemma 2. This yields the monotonicity of G_1 by Lemma 1. Clearly, $G_1(0)=1$. By l'Hôpital's Rule, we get $G_1(1^-)=\infty$.

Next, let $G_5(r)=\cos(r'\mathcal{K})$ and $G_6(r)=r^2$. Then $G_2(r)=\frac{G_5(r)}{G_6(r)}$, $G_5(0)=G_6(0)=0$, and

$$\frac{G'_5(r)}{G'_6(r)}=\frac{\mathcal{K}-\varepsilon}{2r^2}\cdot G_1(r), \tag{16}$$

which is a product of two positive and strictly increasing functions by [6, Lemma 5.2(3)] and by the monotonicity of G_1. The monotonicity of G_2 now follows from Lemma 1.

The limiting value $G_2(1^-)=1$ is clear. By l'Hôpital's Rule and [6, Lemma 5.2(3)], $G_2(0^+)=\frac{\pi}{8}$.

(2) Let $G_7(r)=r'\log\left(\frac{4}{r'}\right)$. Then $H(r)=\frac{G_3(r)}{G_7(r)}$, $G_3(1^-)=G_7(1^-)=0$, and

$$\frac{G'_3(r)}{G'_7(r)}=G_2(r)\cdot\frac{\mathcal{K}-\varepsilon}{\log\left(\frac{4}{r'}\right)-1}, \tag{17}$$

whick is strictly increasing on (0,1) by Part (1) and Lemma 3(3). Hence the monotonicity of H follows from Lemma1.

Clearly, $H(0)=\frac{1}{\log 4}$. By l'Hôpital's Rule, (17), Lemma 3(3) and by Part (1), $H(1^-)=1$, the double inequality(14) holds by the monotonicity of H and Part(1).

Remark. It is clear that in Theorem 2, Part (2) improves the property of G_1 in Part(1). On the other hand, by Theorem 2, we have

$$\mathcal{K}(r)>\frac{[\arccos(r^2)]}{r'},\ \mathcal{K}(r)>\frac{\{\arcsin[P(r)]\}}{r'}>\frac{[\arcsin(r')]}{r'}$$

for all $r\in(0,1)$. It is easy to verify that for all $r\in(0,1)$,

$$\arcsin(r')<\arccos(r^2).$$

Moreover, by elementary method, one can show that there exists a unique number $r_0\in(0,1)$ such that for $r\in(r_0,1)$,

$$\arcsin[P(r)]\begin{cases}<\arccos(r^2), & \text{for } r\in(0,\ r_0),\\ =\arccos(r^2), & \text{for } r=r_0,\\ >\arccos(r^2), & \text{for } r\in(r_0,1).\end{cases} \tag{18}$$

From inequalities (5), (7), (12) and (14), one can obtain the following corollary.

Corollary. For all $r\in(0,1)$, $K\in(1,\infty)$, $\alpha=\frac{(\log 16)}{\pi}$, $A(r)=\arccos\left(\frac{\pi r^2}{8}\right)$ and $B(r)=(1-r)\arcsin\left[\left(\frac{r'}{\log 4}\log\left(\frac{4}{r'}\right)\right]\right.$,

$$\varphi_K(r)<r^{\frac{1}{K}}\exp\left\{\alpha\left[1-\left(\frac{1}{K}\right)\right]r'\mathcal{K}\right\}<r^{\frac{1}{K}}\exp\left\{\alpha\left[1-\left(\frac{1}{K}\right)\right]A(r)\right\}$$

and

$$\varphi_{\frac{1}{K}}(r)<r^K\exp\{\alpha(1-K)(1-r)r'\mathcal{K}\}<r^K\exp\{\alpha(1-K)B(r)\}.$$

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完全椭圆积分$\mathcal{K}(r)$之由三角函数给出的界及其应用

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摘　要： 文章获得了第一类完全椭圆积分$\mathcal{K}(r)$的由正弦和余弦函数给出的上下界与 Hübner 上界函数的一种估计。而且，运用这些结果获得了在拟共形理论等领域中有着重要地位的 Hersch-Pfluger 偏差函数 $\varphi_K(r)$的一类估计。

关键词： 估计不等式；完全椭圆积分；Hersch-Pfluger 偏差函数；拟共形理论

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