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Gamma 函数的单调性与凹凸性

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摘　要：设 $\alpha\in\mathbf{R}$，$\beta\geqslant 0$，$x\in(0,\infty)$，$f_{\alpha,\beta,\pm1}(x)\equiv\left(\dfrac{\mathrm{e}^x\Gamma(x+\beta)}{x^{x+\beta-\alpha}}\right)^{\pm1}$，获得了函数 $f_{\alpha,\beta,+1}$ 的几何凹凸性，从而推广了 $f_{0,0,+1}$ 的相应已知结果；将函数 $f_{\alpha,\beta,\pm1}$ 中的指数函数替换为双曲函数，通过研究其单调性与对数凹凸性，获得了关于 gamma 函数的一些不等式，从而改进了 gamma 函数相应的已知不等式。

关键词：gamma 函数；单调性；几何凹凸性；对数凹凸性；不等式

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Monotonicity and convexity properties for the gamma function

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Abstract：For $\alpha\in\mathbf{R}$，$\beta\geqslant 0$ and $x\in(0,\infty)$，let $f_{\alpha,\beta,\pm1}(x)\equiv\left(\dfrac{\mathrm{e}^x\Gamma(x+\beta)}{x^{x+\beta-\alpha}}\right)^{\pm1}$. In this paper，the authors study the geometric convexity of the function $f_{\alpha,\beta,+1}$，which is a generalization of the corresponding known result of $f_{0,0,+1}$. They also study the monotonicity and logarithmic convexity properties of the gamma function combined with the hyperbolic functions instead of the exponential function in $f_{\alpha,\beta,\pm1}$. Moreover，the inequalities derived from these properties improve some related known results of the gamma function.

Key words：gamma function；monotonicity；geometric convexity；logarithmic convexity；inequality

0　引　言

0　Introduction

The classical Euler gamma function defined by

$$\Gamma(x)=\int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t\ \ (\mathrm{Re}\,x>0)$$

is one of the most important functions in analysis and its applications. The psi or digamma function，the logarithmic derivative of the gamma function，and the polygamma functions can be expressed as

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma+\int_0^\infty\frac{\mathrm{e}^{-t}-\mathrm{e}^{-xt}}{1-\mathrm{e}^{-t}}\mathrm{d}t$$

and

$$\psi^{(n)}(x)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(x+k)^{n+1}}\tag{1}$$

for $n=1,2,\cdots$，where $\gamma=0.5772156649\cdots$ is the Euler-Mascheroni constant.

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The asymptotic formulas of $\Gamma(x)$ and $\psi(x)$ are[1]

$$\Gamma(x) \sim e^{-x} x^{x-\frac{1}{2}} \sqrt{2\pi}\left(1+\frac{1}{12x}+\frac{1}{288x^2}+\cdots\right),\ x \to \infty \text{ with } |\arg x| < \pi \tag{2}$$

and

$$\psi(x) \sim \log x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots,\ x \to \infty \text{ with } |\arg x| < \pi \tag{3}$$

The gamma function plays a significant role in special function theory. By studying the monotonicity and convexity properties of the combinations related to the gamma function and elementary functions, some estimates of the gamma function are obtained, see [2−8].

Let $I \subset (0, \infty)$ be an interval and $f: I \to (0, \infty)$ be a continuous real-valued function. We say that f is geometrically convex (concave) on I if one of the following is true:

$$f(\sqrt{x_1 x_2}) \leqslant (\geqslant) \sqrt{f(x_1) f(x_2)}$$

for all $x_1, x_2 \in I$;

$$f\left(\prod_{i=1}^{n} x_i^{\lambda_i}\right) \leqslant (\geqslant) \prod_{i=1}^{n} f^{\lambda_i}(x_i)$$

for all $x_1, x_2, \cdots, x_n \in I$ and $\lambda_1, \lambda_2, \cdots, \lambda_n \geqslant 0$ with $\prod_{i=1}^{n} \lambda_i = 1$, see [7]. We say that f is logarithmically convex (concave), log-convex (concave) for abbreviation, if

$$f\left(\frac{x+y}{2}\right) \leqslant (\geqslant) \sqrt{f(x) f(y)}$$

for all $x, y \in I$, see [9].

A positive function f is said to be logarithmically completely monotonic (LCM) on an interval I if its logarithm $\log f$ satisfies

$$(-1)^n [\log f(x)]^{(n)} \geqslant 0 \tag{4}$$

for all $x \in I$ and $n = 1, 2, \cdots$. Moreover, f is said to be strictly LCM on I if the inequality (4) is strict, see [10]. Clearly, the function f is decreasing and log-convex if f is LCM on I.

For $\alpha \in \mathbf{R}$, $\beta \geqslant 0$ and $x \in (0, \infty)$, let

$$f_{\alpha,\beta,\pm 1}(x) \equiv (e^x h_{\alpha,\beta}(x))^{\pm 1},$$

where

$$h_{\alpha,\beta}(x) \equiv \frac{\Gamma(x+\beta)}{x^{x+\beta-\alpha}}.$$

Clearly, the function $h_{\alpha,\beta}(x)$ is strictly decreasing and log-convex, and $(h_{\alpha,\beta}(x))^{-1}$ is log-convex on $(0, \infty)$ for certain values (α, β) by Theorem 1 in the sequel.

Let

$$D_1 = \{(\alpha, \beta) \mid \alpha \leqslant 0,\ \beta > 0\},$$

$$D_2 = \left\{(\alpha, \beta) \,\middle|\, \alpha \leqslant \frac{1}{2},\ \beta \geqslant 1\right\},$$

$$D_3 = \left\{(\alpha, \beta) \,\middle|\, \alpha \geqslant \beta,\ \beta \geqslant \frac{1}{2}\right\}.$$

It is known that the function

$$f_{0,0,+1}(x) = \frac{e^x \Gamma(x)}{x^x} \tag{5}$$

is geometrically convex on $(0, \infty)$ by Theorem 1.1 in [8]. It is stated in Theorem 3.2 in [3] that the function $f_{\frac{1}{2},0,+1}$ is decreasing and log-convex from $(0, \infty)$ onto $(\sqrt{2\pi}, \infty)$, and the function $f_{1,0,+1}$ is increasing and log-concave from $(0, \infty)$ onto $(1, \infty)$. As a generalization, Theorem 1 and Theorem 2 in

[11] show that the function $f_{\alpha,\beta,+1}$ is LCM on $(0,\infty)$ for $(\alpha,\beta)\in D_2$ and display the necessary and sufficient conditions for the functions $f_{\alpha,0,+1}$ and $f_{\alpha,0,-1}$ to be LCM on $(0,\infty)$. In [12-13], the authors supplemented some results of $f_{\alpha,\beta,+1}$. The function $f_{\alpha,\beta,-1}$ is also LCM for certain values (α,β), see [14-15].

For a more treatment with the LCM property of the gamma function, we refer the reader to [11-15]. We here only list the results of $f_{\alpha,\beta,\pm1}$ related to the topic of the present paper. The following Theorem 1 is from Theorem 1 in [12] and Theorem 2 in [14].

Theorem 1 a) The function $f_{\alpha,\beta,+1}(x)$ is LCM on $(0,\infty)$ for $(\alpha,\beta)\in D_1$.

b) The function $f_{\alpha,\beta,+1}(x)$ is LCM on $(0,\infty)$ if and only if $\alpha\leqslant\frac{1}{2}$ for $\beta\geqslant 1$.

c) The function $f_{\alpha,\beta,-1}(x)$ is LCM on $(0,\infty)$ if and only if $\alpha\geqslant\beta$ for $\beta\geqslant\frac{1}{2}$.

The following inequalities (6), (7) can be easily derived from the monotonicity and logarithmic convexity of the functions in Theorem 1.

For $0<x<y$, the inequality

$$\frac{\Gamma(y+\beta)}{\Gamma(x+\beta)}\leqslant\frac{e^x}{e^y}\frac{y^{y+\beta-\alpha}}{x^{x+\beta-\alpha}} \tag{6}$$

holds for $(\alpha,\beta)\in D_1\cup D_2$, and the inequality (6) is reversed for $(\alpha,\beta)\in D_3$.

For $x, y>0$, the inequality

$$\frac{\Gamma\left(\frac{x+y}{2}+\beta\right)}{\sqrt{\Gamma(x+\beta)\Gamma(y+\beta)}}\leqslant\frac{\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}+\beta-\alpha}}{x^{\frac{x+\beta-\alpha}{2}}y^{\frac{y+\beta-\alpha}{2}}} \tag{7}$$

holds for $(\alpha,\beta)\in D_1\cup D_2$, and the inequality (7) is reversed for $(\alpha,\beta)\in D_3$. The equalities are true if and only if $x=y$.

In this paper, we generalize the geometric convexity of $f_{0,0,+1}$, see also (5), to $f_{\alpha,\beta,+1}$. We continue the investigation on the monotonicity and logarithmic convexity for the gamma function combined with the hyperbolic functions instead of the exponential function in $f_{\alpha,\beta,\pm1}$. Moreover, the inequalities we obtain from Theorem 3 are all better than the inequalities (6), (7) and their inversed ones in some extent.

Our results are stated as follows.

Theorem 2 The function $g_1(x)\equiv f_{\alpha,\beta,+1}(x)$ is geometrically convex on $(0,\infty)$ for $(\alpha,\beta)\in D_4\cup D_5$, where

$$D_4=\{(\alpha,\beta)\mid\alpha\in R,\ \beta=0\},\ D_5=\{(\alpha,\beta)\mid\alpha\in R,\ \beta\geqslant 1\}.$$

Corollary 1 For $x, y>0$, the inequalities

$$\frac{e^x}{e^y}\frac{y^{y+\beta-\alpha}}{x^{x+\beta-\alpha}}\left(\frac{y}{x}\right)^{x(\psi(x+\beta)-\log x)-\beta+\alpha}\leqslant\frac{\Gamma(y+\beta)}{\Gamma(x+\beta)}\leqslant\frac{e^x}{e^y}\frac{y^{y+\beta-\alpha}}{x^{x+\beta-\alpha}}\left(\frac{y}{x}\right)^{y(\psi(y+\beta)-\log y)-\beta+\alpha}$$

hold for $(\alpha,\beta)\in D_4\cup D_5$.

Theorem 3 Let

$$D_{11}=\{(\alpha,\beta)\mid\alpha\leqslant -1,\ \beta>0\},$$

$$D_{22}=\left\{(\alpha,\beta)\,\middle|\,\alpha\leqslant-\frac{1}{2},\ \beta\geqslant 1\right\},$$

$$D_{33}=\left\{(\alpha,\beta)\,\middle|\,\alpha\geqslant\beta+1,\ \beta\geqslant\frac{1}{2}\right\}.$$

a) The function $g_2(x)\equiv\sinh x\cdot h_{\alpha,\beta}(x)$ is strictly decreasing and strictly log-convex on $(0,\infty)$ for $(\alpha,\beta)\in D_{11}\cup D_{22}$.

b) The function $g_3(x)\equiv\cosh x\cdot h_{\alpha,\beta}(x)$ is strictly increasing and strictly log-concave on $(0,\infty)$ for $(\alpha,\beta)\in D_{33}$.

c) The function $g_4(x)\equiv\sinh x\cdot(h_{\alpha,\beta}(x))^{-1}$ is not decreasing on $(0,\infty)$ for $(\alpha,\beta)\in$

$\{(\alpha,\beta)\mid\alpha\in\mathbf{R},\ \beta\geqslant 0\}$ and is strictly log-convex on $(0,\infty)$ for $(\alpha,\beta)\in D_{33}$.

Theorem 3 leads to the following two corollaries.

Corollary 2 a) For $0<x<y$, the inequality

$$\frac{\Gamma(y+\beta)}{\Gamma(x+\beta)}<\frac{\sinh x}{\sinh y}\frac{y^{y+\beta-\alpha}}{x^{x+\beta-\alpha}} \tag{8}$$

holds for $(\alpha,\beta)\in D_{11}\cup D_{22}$.

b) For $0<x<y$, the inequality

$$\frac{\Gamma(y+\beta)}{\Gamma(x+\beta)}>\frac{\cosh x}{\cosh y}\frac{y^{y+\beta-\alpha}}{x^{x+\beta-\alpha}} \tag{9}$$

holds for $(\alpha,\beta)\in D_{33}$.

Corollary 3 a) For $x,y>0$, the inequality

$$\frac{\Gamma\left(\frac{x+y}{2}+\beta\right)}{\sqrt{\Gamma(x+\beta)\Gamma(y+\beta)}}\leqslant\frac{\sqrt{\sinh x\sinh y}}{\sinh\frac{x+y}{2}}\frac{\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}+\beta-\alpha}}{x^{\frac{x+\beta-\alpha}{2}}y^{\frac{y+\beta-\alpha}{2}}} \tag{10}$$

holds for $(\alpha,\beta)\in D_{11}\cup D_{22}$. The equality is true if and only if $x=y$.

b) For $x,y>0$, the inequality

$$\frac{\Gamma\left(\frac{x+y}{2}+\beta\right)}{\sqrt{\Gamma(x+\beta)\Gamma(y+\beta)}}\geqslant\max\left\{\frac{\sqrt{\cosh x\cosh y}}{\cosh\frac{x+y}{2}},\frac{\sinh\frac{x+y}{2}}{\sqrt{\sinh x\sinh y}}\right\}\frac{\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}+\beta-\alpha}}{x^{\frac{x+\beta-\alpha}{2}}y^{\frac{y+\beta-\alpha}{2}}}=\frac{\sinh\frac{x+y}{2}}{\sqrt{\sinh x\sinh y}}\frac{\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}+\beta-\alpha}}{x^{\frac{x+\beta-\alpha}{2}}y^{\frac{y+\beta-\alpha}{2}}} \tag{11}$$

holds for $(\alpha,\beta)\in D_{33}$. The equality is true if and only if $x=y$.

1　主要结果的证明

1　Proofs of main results

As for prerequisites, the reader is expected to be familiar with some formulas of the hyperbolic functions, which are used in the proofs of our results. In particular, we have the following derivative formulas:

$$(\log\sinh x)'=\coth x,\ (\coth x)'=-\operatorname{csch}^2 x,$$
$$(\log\cosh x)'=\tanh x,\ (\tanh x)'=\operatorname{sech}^2 x.$$

The equivalent assertions of geometric convexity are also needed in the proof of Theorem 2. The following Lemma 1 is from [7, Theorem A,C].

Lemma 1 Let $I\subset(0,\infty)$ be an interval. If $f:I\to(0,\infty)$ is a differentiable real-valued function, then the following assertions are equivalent:

a) The function f is geometrically convex (concave) on I;

b) The function $g(x)\equiv\dfrac{xf'(x)}{f(x)}$ is increasing (decreasing) on I;

c) The function f verifies the inequalities

$$\left(\frac{y}{x}\right)^{\frac{xf'(x)}{f(x)}}\leqslant(\geqslant)\frac{f(y)}{f(x)}\leqslant(\geqslant)\left(\frac{y}{x}\right)^{\frac{yf'(y)}{f(y)}},\ \forall x,y\in I.$$

Now we are in a position to prove Theorem 2 and Theorem 3.

Proof of Theorem 2 It is easy to obtain that

$$\left(x\frac{g_1'(x)}{g_1(x)}\right)'=\psi(x+\beta)+x\psi'(x+\beta)-\log x-1.$$

Let $t=x+\beta$ and

$$h_1(t)\equiv\psi(t)+(t-\beta)\psi'(t)-\log(t-\beta)-1,\ t\in(\beta,\infty).$$

By the formula (1), we have

$$\begin{aligned}h'_1(t)&=2\psi'(t)+(t-\beta)\psi''(t)-\frac{1}{t-\beta}=\sum_{k=1}^{\infty}\frac{2(k+\beta)}{(k+t)^3}+\frac{2\beta}{t^3}-\frac{1}{t-\beta}\\&<\sum_{k=1}^{\infty}\frac{2(k+\beta)}{(k-1+t)(k+t)(k+1+t)}+\frac{2\beta}{t^3}-\frac{1}{t-\beta}\\&=\sum_{k=1}^{\infty}\left(\frac{k+\beta}{(k-1+t)(k+t)}-\frac{k+\beta}{(k+t)(k+1+t)}\right)+\frac{2\beta}{t^3}-\frac{1}{t-\beta}\\&=\frac{\beta+1}{t(t+1)}+\sum_{k=1}^{\infty}\frac{1}{(k+t)(k+1+t)}+\frac{2\beta}{t^3}-\frac{1}{t-\beta}\\&=\frac{\beta+1}{t(t+1)}+\sum_{k=1}^{\infty}\left(\frac{1}{k+t}-\frac{1}{k+1+t}\right)+\frac{2\beta}{t^3}-\frac{1}{t-\beta}\\&=\frac{\beta+1}{t(t+1)}+\frac{1}{t+1}+\frac{2\beta}{t^3}-\frac{1}{t-\beta}\\&=\frac{\beta((1-\beta)t^2+2(1-\beta)t-2\beta)}{t^3(t+1)(t-\beta)}.\end{aligned}$$

Since $(1-\beta)t^2+2(1-\beta)t-2\beta<0$ on $(\beta,\ \infty)$ for $\beta\geqslant1$, we have $h'_1(t)<0$. Together with $h'_1(t)<0$ for $\beta=0$, we get $h_1(t)$ is decreasing on $(\beta,\ \infty)$ for $\beta=0$ or $\beta\geqslant1$. By the formula (3), it follows that

$$\begin{aligned}h_1(t)&>\lim_{t\to\infty}(\psi(t)+(t-\beta)\psi'(t)-\log(t-\beta)-1)\\&=\lim_{t\to\infty}\left(\log t+O\left(\frac{1}{t}\right)+(t-\beta)\left(\frac{1}{t}+O\left(\frac{1}{t^2}\right)\right)-\log(t-\beta)-1\right)\\&=0.\end{aligned}$$

Therefore, $x\dfrac{g'_1(x)}{g_1(x)}$ is increasing. By Lemma 1, we have $g_1(x)$ is geometrically convex on $(0,\ \infty)$ for $(\alpha,\ \beta)\in D_4\cup D_5$.

Proof of Theorem 3 a) By logarithmic differentiation, we have

$$[\log g_2(x)]^{(n)}=[\log f_{\alpha+1,\beta,+1}(x)]^{(n)}+[\log h_2(x)]^{(n)},\ n=1,2,$$

where $h_2(\mathrm{x})\equiv\dfrac{\sinh x}{x\mathrm{e}^x}$.

Since

$$\mathrm{e}^{2x}-1>2x\text{ and }x<\sinh x,$$

by logarithmic differentiation, we obtain

$$[\log h_2(x)]'=\coth x-\frac{1}{x}-1=\frac{2}{\mathrm{e}^{2x}-1}-\frac{1}{x}<0$$

and

$$[\log h_2(x)]''=\frac{1}{x^2}-\mathrm{csch}^2x>0\tag{12}$$

Together with Theorem 1 a) and b), we have

$$[\log g_2(x)]'<0\text{ and }[\log g_2(x)]''>0,$$

and hence $g_2(x)$ is strictly decreasing and strictly log-convex on $(0,\ \infty)$ for $(\alpha,\ \beta)\in D_{11}\cup D_{22}$.

b) By logarithmic differentiation, we have

$$[\log g_3(x)]^{(n)}=[\log h_3(x)]^{(n)}-[\log f_{\alpha-1,\beta,-1}(x)]^{(n)},\ n=1,2,$$

where $h_3(x)\equiv\dfrac{x\cosh x}{\mathrm{e}^x}$.

Since

$$e^{2x}+1>2x \text{ and } \frac{\operatorname{csch}^2 x}{\operatorname{sech}^2 x}=\coth^2 x>1,$$

logarithmic differentiation leads to

$$[\log h_3(x)]'=\tanh x+\frac{1}{x}-1=\frac{1}{x}-\frac{2}{e^{2x}+1}>0$$

and

$$[\log h_3(x)]''=\operatorname{sech}^2 x-\frac{1}{x^2}<\operatorname{csch}^2 x-\frac{1}{x^2}<0.$$

Together with Theorem 1 c), we have

$$[\log g_3(x)]'>0 \text{ and } [\log g_3(x)]''<0,$$

and hence $g_3(x)$ is strictly increasing and strictly log-concave on $(0, \infty)$ for $(\alpha, \beta)\in D_{33}$.

c) To obtain a contradiction, suppose that $g_4(x)$ is decreasing on $(0, \infty)$. By logarithmic differentiation, we have

$$[\log g_4(x)]'=\log x+\frac{x+\beta-\alpha}{x}-\psi(x+\beta)+\coth x\leqslant 0,$$

which is equivalent to

$$\alpha\geqslant\beta-x(\psi(x+\beta)-\log x)+x+x\coth x.$$

By the formula (3), it follows that

$$\alpha\geqslant\lim_{x\to\infty}\left(\beta-x\left(\log(x+\beta)-\frac{1}{2(x+\beta)}+O\left(\frac{1}{x^2}\right)-\log x\right)+x+x\coth x\right)=\infty,$$

which is impossible. Thus $g_4(x)$ is not decreasing on $(0, \infty)$ for $(\alpha, \beta)\in\{(\alpha, \beta)\mid\alpha\in\mathbf{R}, \beta\geqslant 0\}$.

Since

$$[\log g_4(x)]''=[\log f_{\alpha-1,\beta,-1}(x)]''+[\log h_2(x)]'',$$

together with Theorem 1 c) and the formula (12), we have

$$[\log g_4(x)]''>0$$

and hence $g_4(x)$ is strictly log-convex on $(0, \infty)$ for $(\alpha, \beta)\in D_{33}$.

2　注

2　Remarks

Remark 1.　Since e^{-2x} is decreasing from $(0, \infty)$ onto $(0, 1)$, we see that

$$\frac{\sinh x}{\sinh y}=\frac{1-e^{-2x}}{1-e^{-2y}}\frac{e^x}{e^y}<\frac{e^x}{e^y}<\frac{1+e^{-2x}}{1+e^{-2y}}\frac{e^x}{e^y}=\frac{\cosh x}{\cosh y}$$

for $0<x<y$.

Thus the inequality (8) is better than the inequality (6) for $(\alpha, \beta)\in D_{11}\cup D_{22}$ and the inequality (9) is better than the reversed inequality of (6) for $(\alpha, \beta)\in D_{33}$.

Remark 2.　Theorem 3 b) and c) show that

$$\frac{\Gamma\left(\frac{x+y}{2}+\beta\right)}{\sqrt{\Gamma(x+\beta)\Gamma(y+\beta)}}\geqslant\frac{\sqrt{\cosh x\cosh y}}{\cosh\frac{x+y}{2}}\frac{\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}+\beta-\alpha}}{x^{\frac{x+\beta-\alpha}{2}}y^{\frac{y+\beta-\alpha}{2}}}\tag{13}$$

and

$$\frac{\Gamma\left(\frac{x+y}{2}+\beta\right)}{\sqrt{\Gamma(x+\beta)\Gamma(y+\beta)}}\geqslant\frac{\sinh\frac{x+y}{2}}{\sqrt{\sinh x\sinh y}}\frac{\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}+\beta-\alpha}}{x^{\frac{x+\beta-\alpha}{2}}y^{\frac{y+\beta-\alpha}{2}}}\tag{14}$$

for $x, y>0$, $(\alpha, \beta) \in D_{33}$.

By the derivative formulas of hyperbolic functions, it is easy to obtain that $\sinh x$ is strictly log-concave and $\cosh x$ is strictly log-convex on $(0, \infty)$. Then for $x, y>0$,

$$\frac{\sqrt{\sinh x \sinh y}}{\sinh \frac{x+y}{2}} \leqslant 1 \leqslant \frac{\sqrt{\cosh x \cosh y}}{\cosh \frac{x+y}{2}}.$$

Thus the inequality (10) is better than the inequality (7) for $(\alpha, \beta) \in D_{11} \cup D_{22}$ and the inequalities (13) and (14) are both better than the reversed inequality of (7) for $(\alpha, \beta) \in D_{33}$.

Since $\sinh x \cosh x=\frac{1}{2} \sinh 2x$ is strictly log-concave on $(0, \infty)$, we have

$$\frac{\sqrt{\cosh x \cosh y}}{\cosh \frac{x+y}{2}} \leqslant \frac{\sinh \frac{x+y}{2}}{\sqrt{\sinh x \sinh y}},$$

which implies the inequality (11) in Corollary 3.

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