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# Toader 型平均关于其他二元平均的逼近

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**摘　要**：通过研究高斯算术几何平均和其他平均的组合或复合函数的单调性，给出了 Toader 型平均值的由算术几何平均给出的几个精确逼近。作为所得结果的应用，利用 Toader 型平均值与第二类完全椭圆积分的关系，得到第二类完全椭圆积分的新的上下界。这些上下界提供了第一类和第二类完全椭圆积分之间的联系。

**关键词**：Toader 型平均；完全椭圆积分；单调性；精确不等式；逼近
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# On approximation the Toader-type mean by other bivariate means

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**Abstract**: In the article, the authors provide several sharp approximations for the Toader-type mean in terms of Gaussian arithmetic-geometric mean by showing the monotonicity properties for the combinations or compositions of these related means. As the application of the results, by use of the relationship between Toader-type mean and the complete elliptic integral of the second kind, the new bounds for the complete elliptic integral of the second kind were obtained, which provided the relations between the complete integrals of the first kind and second kind.

**Key words**: Toader-type mean; complete elliptic integral; monotonicity; sharp inequalities; approximation

## 0　背景简介

## 0　Introduction

Let $a, b>0$. Toader[1] introduced a mean

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,\mathrm{d}\theta=\begin{cases}\frac{2a}{\pi}\varepsilon\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right), & a>b,\\ a, & a=b,\\ \frac{2b}{\pi}\varepsilon\left(\sqrt{1-\left(\frac{a}{b}\right)^2}\right), & a<b,\end{cases}$$

where

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$$\begin{cases} \varepsilon=\varepsilon(r)=\frac{\pi}{2}\int_0^{\pi/2}\sqrt{1-r^2\sin^2\theta}\,\mathrm{d}\theta, \\ \varepsilon'=\varepsilon'(r)=\varepsilon(r'), \\ \varepsilon(0)=\frac{\pi}{2},\varepsilon(1)=1 \end{cases}$$

for $r\in(0,1)$ and $r'=\sqrt{1-r^2}$ is the complete elliptic integral of the second kind.

The Toader mean $T(a,b)$ is well known in mathematical literature for many years, and it satisfies

$$T(1,r)=\frac{2}{\pi}\varepsilon(r')$$

for $a=1$, $b=r\in(0,1)$. Therefore it cannot be expressed in terms of the elementary transcendental functions.

Let $p\in\mathbf{R}$ and $a,b>0$. Then the p-th power mean $M_p(a,b)$ is defined by

$$M_p(a,b)=\begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p\neq 0, \\ \sqrt{ab}, & p=0. \end{cases}$$

We clearly see that $M_p(a,b)$ is symmetric and homogeneous of degree one with respect to $a$ and $b$, strictly increasing with respect to $p\in\mathbf{R}$ for fixed $a,b>0$ with $a\neq b$. The following well-known inequalities

$$H(a,b)=M_{-1}(a,b)<M_1(a,b)=A(a,b)<M_2(a,b)=Q(a,b)<C(a,b) \tag{1}$$

hold for $a,b>0$ with $a\neq b$, where $H(a,b)=\frac{2ab}{a+b}$, $A(a,b)=\frac{a+b}{2}$, $Q(a,b)=\sqrt{\frac{a^2+b^2}{2}}$ and $C(a,b)=\frac{a^2+b^2}{a+b}$, are the harmonic, arithmetic, quadratic and contraharmonic means of $a$ and $b$, respectively.

The following inequality was conjectured in [2] that

$$M_{3/2}(a,b)<T(a,b)$$

for all $a,b>0$ with $a\neq b$. This inequality was verified in [3-4] respectively. Later in [5], the authors proved the sharp inequality

$$T(a,b)<M_{\ln 2/\ln(\pi/2)}(a,b)$$

for all $a,b>0$ with $a\neq b$. The constants $3/2$ and $\ln 2/\ln(\pi/2)=1.53\cdots$ are best possible.

The classical Gaussian arithmetic-geometric mean $AGM(a,b)$ of two positive real numbers $a$ and $b$ is defined by the common limit of the sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$a_0=a,b_0=b,$$
$$a_{n+1}=\frac{a_n+b_n}{2},b_{n+1}=\sqrt{a_nb_n}.$$

The remarkable Gaussian identity[6] shows that

$$AGM(1,r)=\frac{\pi}{2\kappa(\sqrt{1-r^2})}$$

for all $r\in(0,1)$, where

$$\begin{cases} \kappa=\kappa(r)=\frac{\pi}{2}\int_0^{\pi/2}\frac{1}{\sqrt{1-r^2\sin^2\theta}}\mathrm{d}\theta, \\ \kappa'=\kappa'(r)=\kappa(r'), \\ \kappa(0)=\frac{\pi}{2},\kappa(1)=\infty \end{cases}$$

for $r\in(0,1)$ and $r'=\sqrt{1-r^2}$ is Legendre's complete elliptic integral of the first kind.

Recently, the bounds for the Toader mean $T(a,b)$ and Gaussian arithmetic-geometric means

$AGM(a,b)$ have attracted the interest of many mathematicians, see [7-10]. The following inequalities

$$L(a,b)<AGM(a,b)<A(a,b)<\frac{\pi}{2}L(a,b),$$

$$A^{1/2}(a,b)G^{1/2}(a,b)<AGM(a,b)<\left(\frac{\sqrt{A(a,b)}+\sqrt{G(a,b)}}{2}\right)^2$$

for all $a,b>0$ with $a\neq b$ were established in [11-12], where $G(a,b)=\sqrt{ab}$ and $L(a,b)=\frac{a-b}{\log a-\log b}$ are geometric and logarithmic means, respectively.

Wang et al[13] proved that the constants

$$\alpha_1=2/5,\beta_1=2/\pi\approx 0.6366\cdots \text{ and } \alpha_2=2/3,\beta_2=2\sqrt{2}/\pi\approx 0.9003\cdots$$

are the best possible parameters such that the double inequalities

$$\alpha_1 C(a,b)+(1-\alpha_1)AGM(a,b)<T(a,b)<\beta_1 C(a,b)+(1-\beta_1)AGM(a,b) \tag{2}$$

$$\alpha_2 Q(a,b)+(1-\alpha_2)AGM(a,b)<T(a,b)<\beta_2 Q(a,b)+(1-\beta_2)AGM(a,b) \tag{3}$$

hold for all $a,b>0$ with $a\neq b$.

We note the following fact: if $R_1(a,b),R_2(a,b),R(a,b)$ are means of distinct positive numbers $a$ and $b$ with $R_1(a,b)>R_2(a,b)$, then $R[R_1(a,b),R_2(a,b)]$ is also a mean and satisfies the inequalities

$$R_2(a,b)<R[R_1(a,b),R_2(a,b)]<R_1(a,b).$$

Applying this fact, we can obtain

$$M_0(a,b)=G(a,b)<T(A(a,b),G(a,b))<A(a,b)=M_1(a,b) \tag{4}$$

for all $a,b>0$ with $a\neq b$.

Since

$$AGM(a,b)<T[A(a,b),G(a,b)] \tag{5}$$

it follows from the inequalities (1), (4) and (5) that

$$AGM(a,b)<T[A(a,b),G(a,b)]<A(a.b)<Q(a,b)<C(a,b)$$

for all $a,b>0$ with $a\neq b$.

Motivated by inequalities (2) and (3), in this article we prove several sharp bounds for the Toader-type mean in terms of Gaussian arithmetic-geometric mean by showing the monotonicity properties for the combinations of these related means. Specifically, we work out the optimal parameters $\alpha_1$, $\alpha_2$, $\alpha_3$, $\beta_1$, $\beta_2$, $\beta_3$ on the interval$(0,1)$such that the inequalities

$$\alpha_1 C(a,b)+(1-\alpha_1)AGM(a,b)<T[A(a,b),G(a,b)]<\beta_1 C(a,b)+(1-\beta_1)AGM(a,b),$$

$$\alpha_2 Q(a,b)+(1-\alpha_2)AGM(a,b)<T[A(a,b),G(a,b)]<\beta_2 Q(a,b)+(1-\beta_2)AGM(a,b),$$

$$\alpha_3 A(a,b)+(1-\alpha_3)AGM(a,b)<T[A(a,b),G(a,b)]<\beta_3 A(a,b)+(1-\beta_3)AGM(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

As an application, we provide new estimates for the complete elliptic integral of the second kind: for all $r\in(0,1)$,

$$\frac{\pi^2}{4\kappa(r)}<\varepsilon(r)<\min\left(\frac{1+r^2}{2}+\frac{\pi(\pi-1)}{4\kappa(r)},\sqrt{\frac{1+r^2}{2}}+\frac{\pi(\pi-\sqrt{2})}{4\kappa(r)},1+\frac{\pi(\pi-2)}{4\kappa(r)}\right).$$

## 1 预备知识

## 1 Preliminaries

In this section, we collect some derivative formulas and known monotonicity results for the complete elliptic integrals.

The functions $\kappa(r)$ and $\varepsilon(r)$ satisfy following derivative formulas[14]

$$\frac{d\kappa}{dr}=\frac{\varepsilon-r'^2\kappa}{rr'^2},\frac{d\varepsilon}{dr}=\frac{\varepsilon-\kappa}{r},\frac{d(\varepsilon-r'^2\kappa)}{dr}=r\kappa,\frac{d(\kappa-\varepsilon)}{dr}=\frac{r\varepsilon}{r'^2}.$$

The following monotone form of L'Hopital's rule (MLHR) stated in Theorem 1.25 in [6] is essential in the proof of our main results.

**Lemma 1** (MLHR) Let $-\infty<a<b<+\infty$, and $f,g:[a,b]\to\mathbf{R}$ be continuous functions defined on $[a,b]$ and differentiable in $(a,b)$ with $f(a)=g(a)=0$ or $f(b)=g(b)=0$, and suppose that $g'(x)\neq 0$ on $(a,b)$. If $f'(x)/g'(x)$ is increasing (decreasing, respectively) on $(a,b)$, then so is the function $f(x)/g(x)$.

**Lemma 2** The function

a) $\varphi_1(r)=\dfrac{\kappa(r)-\varepsilon(r)}{r^2}$ is strictly increasing from $(0,1)$ onto $\left(\dfrac{\pi}{4},\infty\right)$.

b) $\varphi_2(r)=\dfrac{\varepsilon(r)-r'^2\kappa(r)}{r^2}$ is strictly increasing from $(0,1)$ onto $\left(\dfrac{\pi}{4},1\right)$.

c) $\varphi_3(r)=\dfrac{\varepsilon(r)^2-r'^2\kappa(r)^2}{r^4}$ is strictly increasing from $(0,1)$ onto $\left(\dfrac{\pi^2}{32},1\right)$.

d) $\varphi_4(r)=\dfrac{\varepsilon(r)^2-r'^2\kappa(r)^2}{\varepsilon(r)-r'^2\kappa(r)}$ is strictly increasing from $(0,1)$ onto $(0,1)$.

e) $\varphi_5(r)=\dfrac{\sqrt{1+r^2}(\varepsilon(r)^2-r'^2\kappa(r)^2)}{r^4}$ is strictly increasing from $(0,1)$ onto $\left(\dfrac{\pi^2}{32},\sqrt{2}\right)$.

f) $\varphi_6(r)=\dfrac{(1+r^2)\varepsilon(r)-r'^2\kappa(r)}{r^4}$ is strictly decreasing from $(0,1)$ onto $(2,+\infty)$.

g) $\varphi_7(r)=\dfrac{(1+r^2)\varepsilon(r)-r'^4\kappa(r)}{r^4}$ is strictly decreasing from $(0,1)$ onto $(2,+\infty)$.

**Proof** Part (a)—(d) can be found in [6]. For part (e), it follows from part (c).

For part (f), a simple computation leads to

$$\varphi_6'(r)=\frac{-4(\varepsilon-r'^2\kappa)-r^2\varepsilon}{r^5}<0.$$

Hence, it is derived that $\varphi_6(r)$ is decreasing on $(0,1)$, and elaborated computation gives

$$\varphi_6(1^-)=2.$$

Let

$$\psi_1(r)=(1+r^2)\varepsilon(r)-r'^2\kappa(r),\psi_2(r)=r^4.$$

Then

$$\varphi_6(r)=\frac{\psi_1(r)}{\psi_2(r)}.$$

A straightforward calculation leads to $\psi_1(0)=\psi_2(0)=0$. Moreover, by differentiation, we have

$$\frac{\psi'_1(r)}{\psi'_2(r)}=\frac{3\varepsilon(r)}{4r^2},$$

which is strictly decreasing, and hence so is the function $\varphi_6(r)$. Using L'Hopital's rule, we obtain

$$\lim_{r\to 0^+}\varphi_6(r)=\lim_{r\to 0^+}\frac{\psi_1(r)}{\psi_2(r)}=+\infty.$$

For part (g), simple computation leads to

$$\varphi'_7(r)=\frac{4(\kappa-\varepsilon)-(7r^2-r^4)\kappa}{r^5}.$$

Let $\phi(r)=4(\kappa-\varepsilon)-(7r^2-r^4)\kappa$, $\phi(0)=0$, $\phi'(r)=(r^2-3)r^2\varepsilon+(3r^2-7)r^2r'^2\kappa$, so that $\phi(r)$ is decreasing on $(0,1)$, together with $\phi(0)=0$, we have $\phi(r)<0$. Hence, it is derived that $\varphi_7(r)$ is decreasing on (0,1), and elaborated computation gives

$$\varphi_7(1^-)=2.$$

Let

$$\psi_3(r)=(1+r^2)\varepsilon(r)-r'^4\kappa(r),\psi_4(r)=r^4.$$

Then

$$\varphi_7(r)=\frac{\psi_3(r)}{\psi_4(r)}.$$

A straightforward calculation gives $\psi_3(0)=\psi_4(0)=0$. Moreover, by differentiation, we have

$$\frac{\psi'_3(r)}{\psi'_4(r)}=\frac{4\varepsilon(r)+(1-3r^2)\kappa(r)}{4r^2}.$$

Using L'Hopital's rule, we obtain

$$\lim_{r\to 0+}\varphi_7(r)=\lim_{r\to 0+}\frac{\psi_3(r)}{\psi_4(r)}=+\infty.$$

## 2 Toader 型平均的逼近

## 2 Approximations of the Toader-type mean

In this section, we present our main results. For convenience, we list some identities for various means

$$T(A(a,b),G(a,b))=\frac{2}{\pi}A(a,b)\varepsilon(r) \tag{6}$$

$$AGM(a,b)=\frac{\pi}{2\kappa(r)}A(a,b) \tag{7}$$

$$C(a,b)=(1+r^2)A(a,b) \tag{8}$$

$$Q(a,b)=\sqrt{1+r^2}A(a,b) \tag{9}$$

where $r=(a-b)/(a+b)\in(0,1)$.

**Theorem 1** The double inequality

$$\alpha_1C(a,b)+(1-\alpha_1)AGM(a,b)<T[A(a,b),G(a,b)]<\beta_1C(a,b)+(1-\beta_1)AGM(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leqslant 0$ and $\beta_1\geqslant 1/\pi$.

**Proof** Without loss of generality, we assume that $a>b>0$. Let $r=(a-b)/(a+b)\in(0,1)$. Using the formulas (6), (7) and (8), we have

$$\frac{T(A(a,b),G(a,b))-AGM(a,b)}{C(a,b)-AGM(a,b)}=\frac{\frac{2}{\pi}A(a,b)\varepsilon(r)-\frac{\pi}{2\kappa(r)}A(a,b)}{(1+r^2)A(a,b)-\frac{\pi}{2\kappa(r)}A(a,b)}$$

$$=\frac{\frac{2}{\pi}\kappa(r)\varepsilon(r)-\frac{\pi}{2}}{(1+r^2)\kappa(r)-\frac{\pi}{2}}=f(r)=\frac{f_1(r)}{f_2(r)},$$

where $f_1(r)=2\kappa(r)\varepsilon(r)/\pi-\pi/2, f_2(r)=(1+r^2)\kappa(r)-\pi/2$. A straightforward calculation leads to $f_1(0)=f_2(0)=0$, and

$$f'_1(r)=\frac{2}{\pi}\frac{\varepsilon(r)^2-r'^2\kappa(r)^2}{rr'^2},f'_2(r)=\frac{(1+r^2)\varepsilon(r)-r'^4\kappa(r)}{rr'^2}.$$

So, we have

$$\frac{f'_1(r)}{f'_2(r)}=\frac{2}{\pi}\,\frac{\varepsilon(r)^2-r'^2\kappa(r)^2}{(1+r^2)\varepsilon(r)-r'^4\kappa(r)}=\frac{2}{\pi}\,\frac{\dfrac{\varepsilon(r)^2-r'^2\kappa(r)^2}{r^4}}{\dfrac{(1+r^2)\varepsilon(r)-r'^4\kappa(r)}{r^4}}.$$

Combining this with Lemma 2 (c) and (g), we see that the function $f_1(r)/f_2(r)$ is strictly increasing on $(0,1)$. Moreover, using L'Hopital's rule, we obtain

$$\lim_{r\to 0^+}\frac{f_1(r)}{f_2(r)}=0,\lim_{r\to 1^-}\frac{f_1(r)}{f_2(r)}=\frac{2}{\pi}\cdot\frac{1}{2}=\frac{1}{\pi} \tag{10}$$

Therefore, Theorem 1 follows from (10) together with the monotonicity of $f(r)$.

**Theorem 2**　The double inequality

$$\alpha_2 Q(a,b)+(1-\alpha_2)AGM(a,b)<T[A(a,b),G(a,b)]<\beta_2 Q(a,b)+(1-\beta_2)AGM(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_2\leqslant 0$ and $\beta_2\geqslant\sqrt{2}/\pi$.

**Proof**　Without loss of generality, we assume that $a>b>0$. Let $r=(a-b)/(a+b)\in(0,1)$. Using the formula (6), (7) and (9), we have

$$\frac{T(A(a,b),G(a,b))-AGM(a,b)}{Q(a,b)-AGM(a,b)}=\frac{\dfrac{2}{\pi}A(a,b)\varepsilon(r)-\dfrac{\pi}{2\kappa(r)}A(a,b)}{\sqrt{1+r^2}A(a,b)-\dfrac{\pi}{2\kappa(r)}A(a,b)}$$

$$=\frac{\dfrac{2}{\pi}\kappa(r)\varepsilon(r)-\dfrac{\pi}{2}}{\sqrt{1+r^2}\kappa(r)-\dfrac{\pi}{2}}=g(r)=\frac{g_1(r)}{g_2(r)},$$

where $g_1(r)=2\kappa(r)\varepsilon(r)/\pi-\pi/2$, $g_2(r)=\sqrt{1+r^2}\,\kappa(r)-\pi/2$. A straightforward calculation leads to $g_1(0)=g_2(0)=0$, and

$$g'_1(r)=\frac{2}{\pi}\,\frac{\varepsilon(r)^2-r'^2\kappa(r)^2}{rr'^2},g'_2(r)=\frac{(1+r^2)\varepsilon(r)-r'^2\kappa(r)}{rr'^2\sqrt{1+r^2}}.$$

So, we have

$$\frac{g'_1(r)}{g'_2(r)}=\frac{2}{\pi}\,\frac{\sqrt{1+r^2}\varepsilon(r)^2-r'^2\kappa(r)^2}{(1+r^2)\varepsilon(r)-r'^2\kappa(r)}=\frac{2}{\pi}\,\frac{\dfrac{\sqrt{1+r^2}\varepsilon(r)^2-r'^2\kappa(r)^2}{r^4}}{\dfrac{(1+r^2)\varepsilon(r)-r'^2\kappa(r)}{r^4}}.$$

Combining this with Lemma 2 (e) and (f), we see that the function $g_1(r)/g_2(r)$ is strictly increasing on $(0,1)$. Moreover, using L'Hopital's rule, we obtain

$$\lim_{r\to 0^+}\frac{g_1(r)}{g_2(r)}=0,\lim_{r\to 1^-}\frac{g_1(r)}{g_2(r)}=\frac{2}{\pi}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{\pi} \tag{11}$$

Therefore, Theorem 2 follows from (11) together with the monotonicity of $g(r)$.

**Theorem 3**　The double inequality

$$\alpha_3 A(a,b)+(1-\alpha_3)AGM(a,b)<T[A(a,b),G(a,b)]<\beta_3 A(a,b)+(1-\beta_3)AGM(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_3\leqslant 0$ and $\beta_3\geqslant 2/\pi$.

**Proof**　Without loss of generality, we assume that $a>b>0$. Let $r=(a-b)/(a+b)\in(0,1)$.

$$\frac{T(A(a,b),G(a,b))-AGM(a,b)}{A(a,b)-AGM(a,b)}=\frac{\dfrac{2}{\pi}A(a,b)\varepsilon(r)-\dfrac{\pi}{2\kappa(r)}A(a,b)}{A(a,b)-\dfrac{\pi}{2\kappa(r)}A(a,b)}=\frac{\dfrac{2}{\pi}\kappa(r)\varepsilon(r)-\dfrac{\pi}{2}}{\kappa(r)-\dfrac{\pi}{2}}=h(r)=\frac{h_1(r)}{h_2(r)},$$

where $h_1(r)=2\kappa(r)\varepsilon(r)/\pi-\pi/2$, $h_2(r)=\kappa(r)-\pi/2$. A straightforward calculation leads to $h_1(0)=h_2(0)=0$, and

$$h'_1(r)=\frac{2}{\pi}\frac{\varepsilon(r)^2-r'^2\kappa(r)^2}{rr'^2},h'_2(r)=\frac{\varepsilon(r)-r'^2\kappa(r)}{rr'^2}.$$

So, we have

$$\frac{h'_1(r)}{h'_2(r)}=\frac{2}{\pi}\frac{\varepsilon(r)^2-r'^2\kappa(r)^2}{\varepsilon(r)-r'^2\kappa(r)}.$$

Combining this with Lemma 2 (d) reveals that the function $h_1(r)/h_2(r)$ is strictly increasing on $(0,1)$. Moreover, using L'Hopital's rule, we obtain

$$\lim_{r\to0+}\frac{h_1(r)}{h_2(r)}=0,\lim_{r\to1-}\frac{h_1(r)}{h_2(r)}=\frac{2}{\pi} \tag{12}$$

Therefore, Theorem 3 follows from (12) together with the monotonicity of $h(r)$.

Let $\alpha=0,\beta_1=1/\pi,\beta_2=\sqrt{2}/\pi,\beta_3=2/\pi$. Then Theorem 1, 2 and 3 imply the following corollary immediately.

**Corollary** The double inequality

$$\frac{\pi^2}{4\kappa(r)}<\varepsilon(r)<\min\left(\frac{1+r^2}{2}+\frac{\pi(\pi-1)}{4\kappa(r)},\sqrt{\frac{1+r^2}{2}}+\frac{\pi(\pi-\sqrt{2})}{4\kappa(r)},1+\frac{\pi(\pi-2)}{4\kappa(r)}\right)$$

holds for all $r\in(0,1)$.

## 3 结 论

## 3 Concluding remarks

In this paper, we study the relation between the Toader-type mean and the Gaussian arithmetic-geometric mean by showing the monotonicity properties for the combinations of these related means. Specifically, we establish the best possible approximations of the Toader mean of the arithmetic mean and geometric mean of two positive numbers by terms of some linear combinations of the Gaussian arithmetic-geometric mean, quadratic mean and contraharmonic mean of these two positive numbers. For some specific values of the numbers, the derived approximations provide some sharp inequalities between the complete elliptic integrals of the first and second kinds.

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