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Some properties of the Cassinian metric

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Abstract：In this paper，the authors obtain the upper and lower bounds of the Cassinian metric in the unit disk and the upper half plane by studying some special formulas of the Cassinian metric. Moreover，they prove the inequalities between the Cassinian metric and the hyperbolic metric in the unit ball and the upper half space.

Key words：Cassinian metric；hyperbolic metric；bounds
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Cassini 度量的一些性质

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摘　要：在单位圆盘和上半平面中，运用所研究的 Cassini 度量的特殊公式来获得 Cassini 度量的上下界；并且证明了 Cassini 度量与双曲度量在单位球和上半空间的比较不等式。

关键词：Cassini 度量；双曲度量；界

0　Introduction

The study of hyperbolic type metrics is a research topic of the geometric function theory. The comparison between the classical hyperbolic metric and new hyperbolic type metrics can reveal the properties of new hyperbolic type metrics well.

In 2009，Ibragimov introduced the Cassinian metric and discussed its geometric properties in [1]. The comparisons between the Cassinian metric and some hyperbolic type metrics were studied in [2-6]. The distortion properties of the Cassinian metric under Möbius transformations of the unit ball (punctured unit ball) onto the unit ball (punctured unit ball) were discussed in [2-3]. The growth of the Cassinian metric under quasiregular mappings of the unit ball onto itself was obtained in [4].

In this paper，we continue the investigation on the basic properties of the Cassinian metric. Specifically，we find some formulas for this metric in some special cases and the bounds of this metric in the unit disk and the upper half plane. In addition，we study the comparisons between the Cassinian metric and the hyperbolic metric.

1　Preliminaries

The hyperbolic metric in the unit ball $B^n=\{z\in\mathbf{R}^n: |z|<1\}$ and the upper half space $H^n=\{x=$

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$(x_1,\cdots,x_n)\in\mathbf{R}^n: x_n>0\}$ are defined as follows. By [7, p.40], for $x, y\in B^n$,

$$\sinh\frac{\rho_{B^n}(x,y)}{2}=\frac{|x-y|}{\sqrt{1-|x|^2}\sqrt{1-|y|^2}},$$

and by [7, p.35], for $x, y\in H^n$,

$$\cosh\rho_{H^n}(x,y)=1+\frac{|x-y|^2}{2x_n y_n}.$$

For a proper subdomain $D\subsetneq\overline{\mathbf{R}^n}$ and for $x, y\in D$, the Cassinian metric c_D is defined as (see [1])

$$c_D(x,y)=\sup_{p\in\partial D}\frac{|x-y|}{|x-p||y-p|}.$$

For a proper subdomain $D\subsetneq\mathbf{R}^n$ and for $x, y\in D$, the scale invariant Cassinian metric $\tilde{\tau}_D$ is defined as (see [8])

$$\tilde{\tau}_D(x,y)=\log\left(1+\sup_{p\in\partial D}\frac{|x-y|}{\sqrt{|x-p||y-p|}}\right).$$

The Euclidean distance from the point x to the boundary ∂D is denoted by $d(x)$. In particular, $d(x)=1-|x|$ in the unit ball and $d(x)=x_n$ in the upper half space.

2 Some basic properties of the Cassinian metric

In [9], the authors studied the formulas in some special cases of $\tilde{\tau}$-metric and gave the estimate for $\tilde{\tau}$-metric in the unit disk and the upper half plane. The definition of c-metric is similar to $\tilde{\tau}$-metric. In this section, we find the special formulas of c-metric in the unit disk and the upper half plane, and combine them with the geometry of c-metric to obtain the bounds of c-metric.

2.1 Formulas for special cases

By [9, Lemma 3.2, Lemma 3.3], it is easy to see that the following two special formulas of c-metric hold in the unit disk.

Lemma 1 Let $x, y\in B^2\setminus\{0\}$ with $|x|=|y|$.

a) If $|x+y|\leqslant\frac{4|x|^2}{1+|x|^2}$, then

$$c_{B^2}(x,y)=\frac{2|x|}{1-|x|^2}.$$

b) If $|x+y|>\frac{4|x|^2}{1+|x|^2}$, then

$$c_{B^2}(x,y)=\frac{|x-y|}{1+|x|^2-|x+y|}.$$

Lemma 2 Let $x, y\in B^2$ with $x=ty$, $t\in\mathbf{R}$ and $|x|\leqslant|y|$.

a) If $t\geqslant 0$, then

$$c_{B^2}(x,y)=\frac{|x-y|}{(1-|x|)(1-|y|)}.$$

b) If $t<0$, then

$$c_{B^2}(x,y)=\frac{|x-y|}{(1+|x|)(1-|y|)}.$$

By [9, Lemma 3.11, Lemma 3.12], we obtain two special formulas of c-metric in the upper half plane.

Lemma 3 Let $x, y\in H^2$ and $d(x)=d(y)=d$.

a) If $|x-y|>2d$, then

$$c_{H^2}(x,y)=\frac{1}{d}.$$

b) If $|x-y| \leqslant 2d$, then

$$c_{H^2}(x,y)=\frac{4|x-y|}{4d^2+|x-y|^2}.$$

Lemma 4 Let $x, y \in H^2$ with $y-x$ orthogonal to ∂H^2. Then

$$c_{H^2}(x,y)=\frac{|x-y|}{d(x)d(y)}.$$

2.2 Bounds of the Cassinian metric

In [1], Ibragimov gave the following upper and lower bounds of c-metric in general domain.

Theorem 1 [1, Lemma 3.4, Lemma 3.5] For all $x, y \in D\backslash\{\infty\}$, we have

$$\frac{|x-y|}{d(x)(d(x)+|x-y|)} \vee \frac{|x-y|}{d(y)(d(y)+|x-y|)} \leqslant c_D(x,y) \leqslant \frac{|x-y|}{d(x)(d(x)-|x-y|)},$$

under an additional condition $|x-y|<d(x)$ for the right-hand side of the inequalities.

According to [9, Theorem 3.6], Lemma 1 and Lemma 2, it is easy to obtain the upper and lower bounds of c-metric in the unit disk.

Theorem 2 For $x, y \in B^2$, we have

$$c_{B^2}(x,y) \geqslant \begin{cases} \dfrac{4\sqrt{|x+y|^2+|x-y|^2}}{4-|x+y|^2-|x-y|^2}, & |x+y|\left(1+\dfrac{4}{|x+y|^2+|x-y|^2}\right) \leqslant 4, \\ \dfrac{4|x-y|}{(2-|x+y|)^2+|x-y|^2}, & |x+y|\left(1+\dfrac{4}{|x+y|^2+|x-y|^2}\right) > 4 \end{cases}$$

and

$$c_{B^2}(x,y) \leqslant \frac{4|x-y|}{(2-|x+y|)^2-|x-y|^2}, |x+y|+|x-y|<2.$$

Remark 1 Let

$$A=\frac{|x-y|}{d(x)(d(x)-|x-y|)} \quad \text{and} \quad B=\frac{4|x-y|}{(2-|x+y|)^2-|x-y|^2}.$$

When $|x-y|<d(x)$, we have

$$|x+y|+|x-y|<|x+y|+d(x) \leqslant |x|+|y|+d(x)=1+|y|<2.$$

Then

$$\begin{aligned} A-B &= \frac{|x-y|}{(1-|x|)(1-|x|-|x-y|)}-\frac{4|x-y|}{(2-|x+y|)^2-|x-y|^2} \\ &= |x-y| \frac{(2|x|-|x+y|+|x-y|)(2-|x+y|-|x-y|+2(1-|x|))}{(1-|x|)(1-|x|-|x-y|)((2-|x+y|)^2-|x-y|^2)}. \end{aligned}$$

If $|x| \geqslant |y|$, then

$$2|x|-|x+y|+|x-y| \geqslant |x|+|y|-|x+y|+|x-y| \geqslant |x-y| \geqslant 0.$$

If $|x|<|y|$, then

$$2|x|-|x+y|+|x-y|>|x|+|y|-|x+y|>0.$$

Hence $A \geqslant B$.

Therefore, the upper bound of c_{B^2} in Theorem 2 is less than that in Theorem 1.

We give the estimate for c-metric in the upper half plane by [9, Theorem 3.14], Lemma 3 and Lemma 4.

Theorem 3 For $x, y \in H^2$, we have

$$c_{H^2}(x,y) \geqslant \begin{cases} \dfrac{2}{d(x)+d(y)}, & |x-y|>d(x)+d(y), \\ \dfrac{4|x-y|}{(d(x)+d(y))^2+|x-y|^2}, & |x-y| \leqslant d(x)+d(y), \end{cases}$$

and

$$c_{H^2}(x,y) \leqslant \frac{4|x-y|}{(d(x)+d(y))^2-|x-y|^2}, \quad |x-y| < d(x)+d(y).$$

Remark 2 Let

$$C = \frac{4|x-y|}{(d(x)+d(y))^2-|x-y|^2}.$$

When $|x-y| < d(x)$, we have

$$\begin{aligned} A - C &= \frac{|x-y|}{d(x)(d(x)-|x-y|)} - \frac{4|x-y|}{(d(x)+d(y))^2-|x-y|^2} \\ &= |x-y| \frac{(d(y)+|x-y|-d(x))(3d(x)+d(y)-|x-y|)}{d(x)(d(x)-|x-y|)((d(x)+d(y))^2-|x-y|^2)}. \end{aligned}$$

Since

$$d(y)+|x-y|-d(x) = y_n + |x-y| - x_n \geqslant |x-y_1e_1| - x_n \geqslant 0,$$

we have $A \geqslant C$.

Therefore, the upper bound of c_{H^2} in Theorem 3 is less than that in Theorem 1.

3 The Cassinian metric and the hyperbolic metric

The comparison between the Cassinian metric and the hyperbolic metric in the unit ball is shown in [2, Theorem 3.1]:

$$\sinh \frac{\rho_{B^n}(x,y)}{2} \leqslant c_{B^n}(x,y),$$

while the upper bound of c-metric with respect to ρ-metric was missing. In this section, we will investigate the upper bound of c_{B^n}. Furthermore, we estimate c-metric in terms of ρ-metric in the upper half space.

The following inequalities will be used in the proofs of the results in this section.

Since

$$2\sinh x = e^x - e^{-x} = \frac{e^{2x}-1}{e^x} \leqslant e^{2x}-1, \quad x \geqslant 0,$$

together with [5, Proof of Theorem 3.6]

$$1 + 2\sinh x = 1 + e^x - e^{-x} \geqslant e^x, \quad x \geqslant 0,$$

we have

$$e^x - 1 \leqslant 2\sinh x \leqslant e^{2x} - 1, \quad x \geqslant 0 \tag{1}$$

By [10, p.23, (2.13)], we have

$$\log(1+x) \leqslant \operatorname{arcsinh} x \leqslant 2\log(1+x), \quad x \geqslant 0 \tag{2}$$

The well-known Bernoulli inequality reads as [5, Proof of Lemma 5.7]

$$\log(1+ax) \geqslant a\log(1+x), \quad 0 < a < 1, x \geqslant 0 \tag{3}$$

By [2, Proof of Lemma 3.6], we have

$$\log(1+x) > \frac{2x}{2+x}, \quad x > 0 \tag{4}$$

Theorem 4 For all $x, y \in B^n$ with $|x| \vee |y| = \lambda$, we have

$$c_{B^n}(x,y) \leqslant \frac{1+\lambda}{1-\lambda} \sinh \frac{\rho_{B^n}(x,y)}{2}.$$

Proof. Without loss of generality, we may assume that $\lambda = |x|$. Then we have

$$\sqrt{(1-|x|^2)(1-|y|^2)} = \sqrt{(1-|x|)(1+|x|)(1-|y|)(1+|y|)} \leqslant (1-|y|)(1+|x|) \tag{5}$$

For $x, y \in B^n$, we have

$$\inf_{p\in\partial B^n}|x-p||y-p|\geqslant(1-|x|)(1-|y|) \tag{6}$$

Now

$$\sinh\frac{\rho_{B^n}(x,y)}{2}=\frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}}\geqslant\frac{|x-y|}{(1-|y|)(1+|x|)}$$
$$\geqslant\frac{1-|x|}{1+|x|}c_{B^n}(x,y)=\frac{1-\lambda}{1+\lambda}c_{B^n}(x,y).$$

This completes the proof.□

By Theorem 4 and (1), we have the following corollary.

Corollary 1 For all $x,y\in B^n$ with $|x|\vee|y|=\lambda$, we have

$$c_{B^n}(x,y)\leqslant\frac{1+\lambda}{2(1-\lambda)}(e^{\rho_{B^n}(x,y)}-1).$$

Theorem 5 For all $x,y\in B^n$ with $|x|\vee|y|=\lambda$, we have

$$c_{B^n}(x,y)\leqslant\frac{2+\lambda}{2(1-\lambda)^2}\rho_{B^n}(x,y).$$

Proof. By (5), we have

$$\sqrt{(1-|x|^2)(1-|y|^2)}\leqslant 1+\lambda.$$

By (6), we have

$$\inf_{p\in\partial B^n}|x-p||y-p|\geqslant(1-\lambda)^2.$$

By (2)—(4), we have

$$\rho_{B^n}(x,y)=2\operatorname{arcsinh}\frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}}\geqslant 2\log\left(1+\frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}}\right)$$
$$\geqslant|x-y|\log\left(1+\frac{2}{\sqrt{(1-|x|^2)(1-|y|^2)}}\right)\geqslant\frac{2|x-y|}{1+\sqrt{(1-|x|^2)(1-|y|^2)}}$$
$$\geqslant\frac{2|x-y|}{2+\lambda}\geqslant\frac{2(1-\lambda)^2}{2+\lambda}c_{B^n}(x,y).$$

This completes the proof. □

Remark 3 Let

$$f(x,\lambda)=\frac{1+\lambda}{1-\lambda}\sinh\frac{x}{2}\quad\text{and}\quad g(x,\lambda)=\frac{2+\lambda}{2(1-\lambda)^2}x.$$

The comparison between $f(x,\lambda)$ and $g(x,\lambda)$ is changing as x and λ take different values, see figure 1.

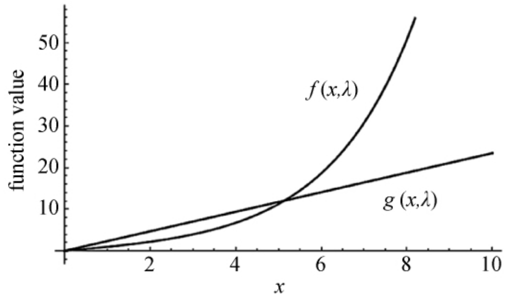

Figure 1　The graphs of $f(x,\lambda)$ and $g(x,\lambda)$, when $\lambda=0.3$.

Theorem 6 For all $x,y\in H^n$ with $|x|\vee|y|=\lambda$ and $d(x)\wedge d(y)=\mu$, we have

$$\frac{2}{3\lambda}\sinh\frac{\rho_{H^n}(x,y)}{2}\leqslant c_{H^n}(x,y)\leqslant\frac{2}{\mu}\sinh\frac{\rho_{H^n}(x,y)}{2}.$$

Proof. By the definition of the hyperbolic metric in the upper half space and the formula $\cosh x=$

$2\sinh^2\frac{x}{2}+1$, we have

$$2\sinh\frac{\rho_{H^n}(x,y)}{2}=\frac{|x-y|}{\sqrt{x_n y_n}}. \tag{7}$$

Without loss of generality, we may assume that $x_n \leqslant y_n$. By the triangle inequality, we have

$$\inf_{p\in\partial H^n}|x-p||y-p|\leqslant x_n(x_n+|x-y|)\leqslant x_n(|x|+|x|+|y|)\leqslant 3\lambda x_n.$$

Then we obtain

$$c_{H^n}(x,y)\geqslant\frac{|x-y|}{3\lambda x_n}\geqslant\frac{1}{3\lambda}\frac{|x-y|}{\sqrt{x_n y_n}}=\frac{2}{3\lambda}\sinh\frac{\rho_{H^n}(x,y)}{2}.$$

This proves the first inequality.

For the proof of the second inequality, we observe that

$$\inf_{p\in\partial H^n}|x-p||y-p|\geqslant x_n y_n,\quad \text{for}\quad x,y\in H^n.$$

By (7), we have

$$c_{H^n}(x,y)\leqslant\frac{|x-y|}{x_n y_n}=\frac{1}{\sqrt{x_n y_n}}\frac{|x-y|}{\sqrt{x_n y_n}}=\frac{2}{\sqrt{x_n y_n}}\sinh\frac{\rho_{H^n}(x,y)}{2}\leqslant\frac{2}{\mu}\sinh\frac{\rho_{H^n}(x,y)}{2}.$$

Hence the inequalities hold.□

By Theorem 6 and (1), we have the following corollary.

Corollary 2 For all $x,y\in H^n$ with $|x|\vee|y|=\lambda$ and $d(x)\wedge d(y)=\mu$, we have

$$\frac{1}{3\lambda}(e^{\frac{\rho_{H^n}(x,y)}{2}}-1)\leqslant c_{H^n}(x,y)\leqslant\frac{1}{\mu}(e^{\rho_{H^n}(x,y)}-1).$$

4 Concluding Remark

In order to better understand the Cassinian metric, it is natural to study the formulas in special cases, the bounds of this metric, and the comparisons between the Cassinian metric and the classical hyperbolic metric. The formulas in the extreme status are helpful to prove sharp inequalities of the Cassinian metric under Möbius transformations. The bounds of the Cassinian metric and the relations between this metric and the hyperbolic metric can be used to study the comparisons between the Cassinian metric and other hyperbolic type metrics.

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