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Bounds and comparison inequalities for the triangular ratio metric

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Abstract：The purpose of this paper is to establish upper and lower bounds for the triangular ratio metric，and to present geometric properties of the triangular ratio metric by comparing it with the distance ratio metric，the hyperbolic metric，the Cassinian metric，and a Gromov hyperbolic metric. In particular，we show the sharpness of the inequalities between the triangular ratio metric and these hyperbolic type metrics.

Key words：triangular ratio metric；hyperbolic type metrics；convex domain

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三角比度量的界和比较不等式

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摘　要：研究了三角比度量在单位圆盘内的上下界，分别证明了三角比度量与距离比度量、双曲度量、Cassini 度量 和 Gromov 双曲度量等四个度量的比较关系。特别地，证明了三角比度量与这些双曲型度量之间不等式的精确性。

关键词：三角比度量；双曲型度量；凸域

0　Introduction

One of the aspects of hyperbolic type metric geometry deals with the comparison of the hyperbolic metric with the hyperbolic type metrics. The invariance and distortion properties of hyperbolic type metrics under conformal maps and quasiconformal maps also play significant roles in geometric function theory.

Recently，the triangular ratio metric was first introduced and studied in [1]. Some basic properties and distortion inequalities of the triangular ratio metric under Möbius transformations were considered in [2-3]. The behavior of this metric under quasiconformal maps was mainly studied in [2].

In this paper，we continue the research of the triangular ratio metric. We give upper and lower bounds for the triangular ratio metric in the unit disk. Additionally，we compare the triangular ratio metric with the distance ratio metric and the Cassinian metric. We also prove sharp inequalities of the triangular ratio metric in terms of the hyperbolic metric and a Gromov hyperbolic metric.

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1 Preliminaries

A domain $G \subset \mathbf{R}^n$ is an open and connected subset of the Euclidean n-space $\mathbf{R}^n$. For $x, y \in G$, the Euclidean distance between x and y is usually denoted by $|x-y|$. The notation $d(x)$ for abbreviation, stands for the distance from the point x to the boundary ∂G of the domain G. For all x, y in the upper half space $H^n = \{(x_1, \cdots, x_n) \in \mathbf{R}^n : x_n > 0\}$, the point $\overline{y} = (y_1, \cdots, y_{n-1}, -y_n)$ is the reflection of $y = (y_1, \cdots y_{n-1}, y_n)$ with respect to ∂H^n.

1.1 The triangular ratio metric

The triangular ratio metric is defined as follows: for a domain $G \subsetneq \mathbf{R}^n$ and $x, y \in G$,

$$s_G(x, y) = \sup_{p \in \partial G} \frac{|x-y|}{|x-p| + |y-p|} \in [0, 1].$$

Clearly, the supremum in the definition is attained at some point $z \in \partial G$.

1.2 The hyperbolic metric

The hyperbolic metrics ρ_{B^n} of the unit ball $B^n = \{z \in \mathbf{R}^n : |z| < 1\}$ is defined as follows [4]:

$$\operatorname{sh} \frac{\rho_{B^n}(x, y)}{2} = \frac{|x-y|}{\sqrt{1-|x|^2}\sqrt{1-|y|^2}},$$

$$\operatorname{th} \frac{\rho_{B^n}(x, y)}{2} = \frac{|x-y|}{\sqrt{|x-y|^2 + (1-|x|^2)(1-|y|^2)}}.$$

1.3 The distance ratio metric

Let $G \subset \mathbf{R}^n$ be a proper open subset of $\mathbf{R}^n$. For all $x, y \in G$, the distance ratio metric j_G is defined as

$$j_G(x, y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right).$$

The above form of the metric j_G, which was first considered in [5], is a slight modification of the original distance ratio metric introduced in [6-7].

By [8, Lemma 2.41(2)] and [9, Lemma 7.56], for all $x, y \in G$,

$$j_G(x, y) \leqslant \rho_G(x, y) \leqslant 2 j_G(x, y),$$

where $G \in \{B^n, H^n\}$.

1.4 The Cassinian metric

For a domain $G \subsetneq \overline{\mathbf{R}}^n$ and $x, y \in G$,

$$c_G(x, y) = \sup_{p \in \partial G} \frac{|x-y|}{|x-p||y-p|}.$$

The term Cassinian metric was introduced by Ibragimov in [10], and the geometry of the Cassinian metric including geodesics, isometries and completeness was first studied there.

1.5 The Gromov hyperbolic metric

For a domain $G \subsetneq \mathbf{R}^n$, the u-metric is defined by

$$u_G(x, y) = 2\log \frac{|x-y| + \max\{d(x), d(y)\}}{\sqrt{d(x)d(y)}}, \quad x, y \in G.$$

By [11, Theorem 6], for $x, y \in B^n$,

$$\rho_{B^n}(x, y) \leqslant u_{B^n}(x, y) \leqslant 3\rho_{B^n}(x, y) \tag{1}$$

Both inequalities are sharp.

2 Bounds of the triangular ratio metric

This section is devoted to looking for upper and lower bounds for the triangular ratio metric in terms of the quantities, in the unit disk. In the special case $|x|=|y|$, we have been able to find an explicit formula for $s_{B^2}(x,y)$.

Lemma 1[12, Theorem 3.1] Let $a=\alpha+i\beta, \alpha,\beta>0$, be a point in the unit disk. Then, we have

$$s_{B^2}(a,\overline{a})=\begin{cases}\dfrac{\beta}{\sqrt{\beta^2+(1-\alpha)^2}}, & \left|a-\left(\dfrac{1}{2},0\right)\right|\leqslant\dfrac{1}{2},\\ |a|, & \left|a-\left(\dfrac{1}{2},0\right)\right|>\dfrac{1}{2}.\end{cases}$$

Lemma 2 Let $x,y\in B^2$ with $x=ty$, $t\in[0,1]$.

a) If $t\geqslant 0$, then

$$s_{B^2}(x,y)=\frac{|x-y|}{2-|x|-|y|}.$$

b) If $t<0$, then

$$s_{B^2}(x,y)=\frac{|x-y|}{2+|x|-|y|}.$$

Proof: From the definition of the triangular ratio metric it follows that,

if $t\geqslant 0$, then

$$s_{B^2}(x,y)=\frac{|x-y|}{(1-|x|)+(1-|y|)}=\frac{|x-y|}{2-|x|-|y|};$$

if $t<0$, then

$$s_{B^2}(x,y)=\frac{|x-y|}{(1-|y|)+(1+|x|)}=\frac{|x-y|}{2+|x|-|y|}.\square$$

By Lemma 1, we shall give an explicit formula for $s_{B^2}(x,y)$ in the case when $|x|=|y|<1$.

Lemma 3 Let $x,y\in B^2\backslash\{0\}, x=(x_1,x_2)$ and $y=(x_1,-x_2)$.

a) If $|x_1|<|x|^2$, then

$$s_{B^2}(x,y)=|x|.$$

b) If $|x_1|\geqslant|x|^2$, then

$$s_{B^2}(x,y)=\frac{|x_2|}{\sqrt{|x|^2-2|x_1|+1}}.$$

Proof: Without loss of generality, we may assume that $x_1,x_2>0$. If $x_1<|x|^2$, then $\left|x_1-\left(\frac{1}{2},0\right)\right|^2+{x_2}^2>\frac{1}{4}\Rightarrow\left|x-\left(\frac{1}{2},0\right)\right|>\frac{1}{2}$. By Lemma 1, we have

$$s_{B^2}(x,y)=|x|.$$

Similarly, the condition $x_1\geqslant|x|^2$ implies $\left|x-\left(\frac{1}{2},0\right)\right|\leqslant\frac{1}{2}$, thus

$$s_{B^2}(x,y)=\frac{x_2}{\sqrt{|x|^2-2x_1+1}}.\square$$

Theorem 1 Let $x,y,x',y',x'',y''\in B^2$ with $x'=\frac{x+y}{2}-\frac{|x-y|}{2}\xi$, $y'=\frac{x+y}{2}+\frac{|x-y|}{2}\xi$, $x''=\frac{x+y}{2}-\frac{|x-y|}{2}\zeta$, $y''=\frac{x+y}{2}+\frac{|x-y|}{2}\zeta$, where

$$\xi=\begin{cases}\dfrac{x+y}{|x+y|}, & x+y\neq 0,\\ e_1, & x+y=0.\end{cases}$$

and $\zeta=i\xi$. Then

$$s_{B^2}(x'',y'')\leqslant s_{B^2}(x,y)\leqslant s_{B^2}(x',y'),$$

where

$$s_{B^2}(x,y)\leqslant\frac{|x-y|}{2-|x+y|}\tag{2}$$

and

$$s_{B^2}(x,y)\geqslant\begin{cases}\dfrac{\sqrt{|x+y|^2+|x-y|^2}}{2}, & |x+y|<\dfrac{|x+y|^2+|x-y|^2}{2},\\ \dfrac{|x-y|}{\sqrt{(|x+y|-2)^2+|x-y|^2}}, & |x+y|\geqslant\dfrac{|x+y|^2+|x-y|^2}{2}\end{cases}\tag{3}$$

Proof: Since the result is trivial for the case $x=-y$ by symmetry. Then we may assume that $x\neq -y$.

We easily see that the extremal ellipse with foci x,y is larger than the extremal ellipse with foci x', y', hence

$$s_{B^2}(x,y)\leqslant s_{B^2}(x',y').$$

To prove inequality (2), we have

$$x'=\frac{x+y}{2|x+y|}(|x+y|-|x-y|)\text{ and }y'=\frac{x+y}{2|x+y|}(|x+y|+|x-y|).$$

It is easy to see that $x'=ty'$, $t\in[0,1]$ and $|x'|\leqslant|y'|$.

Case (ⅰ). If $t=0$, then $|x'|=0$ and $|x+y|=|x-y|$. By Lemma 2 a), we have

$$s_{B^2}(x',y')=s_{B^2}(0,y')=\frac{|y'|}{2-|y'|}=\frac{|x-y|}{2-|x+y|}.$$

Case (ⅱ). If $t>0$, then $|x'|=\dfrac{|x+y|-|x-y|}{2}$ and $|y'|=\dfrac{|x+y|+|x-y|}{2}$. By Lemma 2 a), we have

$$s_{B^2}(x,y)\leqslant s_{B^2}(x',y')=\frac{|x'-y'|}{2-|x'|-|y'|}=\frac{|x-y|}{2-|x+y|}.$$

Case (ⅲ). If $t<0$, then $|x'|=\dfrac{|x-y|-|x+y|}{2}$ and $|y'|=\dfrac{|x-y|+|x+y|}{2}$. By Lemma 2 b), we have

$$s_{B^2}(x,y)\leqslant s_{B^2}(x',y')=\frac{|x'-y'|}{2+|x'|-|y'|}=\frac{|x-y|}{2-|x+y|}.$$

To prove inequalities (3), we have

$$x''=\frac{x+y}{2|x+y|}(|x+y|-i|x-y|)\quad\text{and}\quad y''=\frac{x+y}{2|x+y|}(|x+y|+i|x-y|).$$

Case (ⅳ). If $|x+y|<\dfrac{|x+y|^2+|x-y|^2}{2}$, then $\dfrac{|x+y|}{2}<|x''|^2$. By Lemma 3 a), we have

$$s_{B^2}(x,y)\geqslant s_{B^2}(x'',y'')=|x''|=\frac{\sqrt{|x+y|^2+|x-y|^2}}{2}.$$

Case (ⅴ). If $|x+y|\geqslant\dfrac{|x+y|^2+|x-y|^2}{2}$, then $\dfrac{|x+y|}{2}\geqslant|x''|^2$. By Lemma 3 b), we have

$$s_{B^2}(x,y)\geqslant s_{B^2}(x'',y'')=\frac{|x-y|}{\sqrt{(|x+y|-2)^2+|x-y|^2}}.\square$$

3 Comparisons with other related metric

In this section, we compare the triangular ratio metric with the distance ratio metric in the convex domain. We also study geometric properties of the triangular ratio metric by comparing it with the hyperbolic metric, the Cassinian metric, and a Gromov hyperbolic metric. In particular, we show the sharpness of the inequalities between the triangular ratio metric and these hyperbolic metrics.

Theorem 2 Let G be a proper convex subdomain of $\mathbf{R}^n$. Let $x,y\in G$ and $t=e^{j_G(x,y)}-1$. Then

$$\frac{t}{t+2}\leqslant s_G(x,y)\leqslant \frac{t}{\sqrt{4+t^2}}.$$

Both inequalities are sharp.

Proof: Without loss of generality, we may assume that $d(x)\leqslant d(y)$. Then $t=\frac{|x-y|}{d(x)}$. Choose $z\in \partial G$ such that $d(x)=|x-z|$. Let $y'\in \mathrm{ray}(x,x-z)=\{x+t(x-z):t>0\}$ with $|y-x|=|y'-x|$. Let $H_G, x\in H_G$, be the half space whose boundary is orthogonal to $[x,z]$ at the point z. It is easy to see that $G\subset H_G$. Let B_{xy} be the convex hull of $B^n(x,d(x))\cup B^n(y,d(y))$. By the domain monotonicity property of s_G, we have

$$s_{H_G}(x,y)\leqslant s_G(x,y)\leqslant s_{B_{xy}}(x,y),$$

where

$$s_{H_G}(x,y)\geqslant s_{H_G}(x,y')=\frac{|x-y|}{d(x)+d(y')}=\frac{|x-y|}{|x-y|+2d(x)}=\frac{t}{t+2},$$

and

$$s_{B_{xy}}(x,y)=\frac{|x-y|}{2\sqrt{d^2(x)+(|x-y|/2)^2}}=\frac{t}{\sqrt{4+t^2}}.$$

For the sharpness of the left-hand side of the inequality, we consider the domain H^n and two points $x,y\in H^n$ with the line $L(x,y-x)$ passing through x with direction vector $y-x$ which is perpendicular to the boundary ∂H^n.

$$s_{H^n}(x,y)=\frac{|x-y|}{|x-y|+2d(x)}=\frac{t}{t+2}.$$

For the sharpness of the right-hand side of the inequality, we consider the domain H^n and two points $x,y\in H^n$ with the line $L(x,y-x)$ parallel to the boundary ∂H^n.

$$s_{H^n}(x,y)=\frac{|x-y|}{2\sqrt{d^2(x)+(|x-y|/2)^2}}=\frac{t}{\sqrt{t^2+4}}.\square$$

Theorem 3[3, Lemma 2.6] For $x,y\in B^n$,

$$\mathrm{th}\frac{\rho_{B^n}(x,y)}{4}\leqslant s_{B^n}(x,y)\leqslant th\frac{\rho_{B^n}(x,y)}{2}.$$

Both inequalities are sharp.

Proof: For the inequalities see [3, Lemma 2.6].

For the sharpness of the left-hand side of inequalities, let $x=te_1$ and $y=(t+(1-t)^2)e_1$ with $t\in(0,1)$. Then

$$\mathrm{th}\frac{\rho_{B^n}(x,y)}{2}=\frac{(1-t)^2}{\sqrt{(1-t)^4+(1-t^2)(1-(t+(1-t)^2)^2)}}=\frac{1-t}{1+t^2}\tag{4}$$

By [8, (2.29)] and (4),

$$\mathrm{th}\frac{\rho_{B^n}(x,y)}{4}=\frac{\mathrm{th}(\rho_{B^n}(x,y)/2)}{1+\sqrt{1-\mathrm{th}^2(\rho_{B^n}(x,y)/2)}}=\frac{(1-t)/(1+t^2)}{1+\sqrt{1-(1-t)^2/(1+t^2)^2}}=\frac{1-t}{1+t^2+\sqrt{t^4+t^2+2t}}.$$

Since

$$s_{B^n}(x,y)=\frac{(1-t)^2}{1-t+1-(t+(1-t)^2)}=\frac{1-t}{1+t},$$

then we have

$$\lim_{t\to 0}\frac{\operatorname{th}(\rho_{B^n}(x,y)/4)}{s_{B^n}(x,y)}=\lim_{t\to 0}\frac{(1-t)/(1+t^2+\sqrt{t^4+t^2+2t})}{(1-t)/(1+t)}=\lim_{t\to 0}\frac{1+t}{1+t^2+\sqrt{t^4+t^2+2t}}=1.$$

For the sharpness of the right-hand side of inequalities, let $x=-y=te_1$ with $t\in(0,1)$. Then

$$\lim_{t\to 1}\frac{s_{B^n}(x,y)}{\operatorname{th}(\rho_{B^n}(x,y)/2)}=\lim_{t\to 1}\frac{2t/((1-t)+(1+t))}{2t/\sqrt{4t^2+(1-t^2)^2}}=\lim_{t\to 1}\frac{1+t^2}{2}=1.\square$$

Theorem 4　For $x,y\in B^n$, we have

$$\operatorname{th}\frac{u_{B^n}(x,y)}{12}\leqslant s_{B^n}(x,y)\leqslant \operatorname{th}\frac{u_{B^n}(x,y)}{2}.$$

Both inequalities are sharp.

Proof: These two inequalities are easily derived from inequalities (1) and Theorem 3.

For the sharpness of the left-hand side of the inequalities, let $x=te_1$ and $y=(t+(1-t)^2)e_1$ with $t\in(0,1)$. Since

$$u_{B^n}(x,y)=2\log\frac{(1-t)^2+1-t}{\sqrt{(1-t)(1-t-(1-t)^2)}}=\log\frac{(2-t)^2}{t},$$

then we have

$$\lim_{t\to 0}\frac{s_{B^n}(x,y)}{\operatorname{th}(u_{B^n}(x,y)/12)}=1.$$

Therefore, the left-hand side of the inequalities is sharp.

For the sharpness of the right-hand side of the inequalities, let $x=-y=te_1$ with $t\in(0,1)$. By [11, Theorem 6] and Theorem 3, we have

$$\lim_{t\to 1}\frac{s_{B^n}(x,y)}{\operatorname{th}(u_{B^n}(x,y)/2)}=1.$$

Therefore, the right-hand side of the inequalities is sharp. $\square$

Theorem 5　Let $x,y\in H^2$ and $x\in B^2\left(\frac{y+\overline{y}}{2},d(y)\right)$ or $y\in B^2(\frac{x+\overline{x}}{2},d(x))$, $r=\min\{d(x),d(y)\}$, we have

$$\frac{r}{2}c_{H^2}(x,y)\leqslant s_{H^2}(x,y)\leqslant rc_{H^2}(x,y).$$

Proof: Without loss of generality, we may assume that $d(x)\leqslant d(y)$. We have

$$s_{H^2}(x,y)=\frac{|x-y|}{|\overline{x}-y|}.$$

Let z be a point on the segment $[x,\overline{x}]\cap\partial H^2$, then $c_{H^2}(x,y)\geqslant\frac{|x-y|}{d(x)|y-z|}$ and

$$\frac{s_{H^2}(x,y)}{c_{H^2}(x,y)}\leqslant\frac{|y-z|}{|\overline{x}-y|}d(x)\leqslant r.$$

Hence the second inequality holds. For the first inequality,

$$s_{H^2}(x,y)=\frac{|x-y|}{|\overline{x}-y|}\geqslant\frac{|x-y|}{2d(x)}=\frac{d(x)}{2}\frac{|x-y|}{d(x)d(y)}\geqslant\frac{r}{2}c_{H^2}(x,y).$$

Thus,

$$\frac{r}{2}c_{H^2}(x,y)\leqslant s_{H^2}(x,y)\leqslant rc_{H^2}(x,y).\square$$

4　Concluding remark

Several authors have studied the triangular ratio metric in subdomains of the complex plane and

Euclidean n-space, and proved various inequalities and the behavior of this metric under Möbius transformations, bilipschtiz maps and quasiconformal maps. In order to understand the geometric properties of this metric, we establish upper and lower bounds for the triangular ratio metric and compare it with some hyperbolic type metrics. The results will play an important role in the study of the inclusion relations of the related metric balls.

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