浙江理工大学学报，2020，43(6)：852-857
Journal of Zhejiang Sci-Tech University
DOI：10.3969/j.issn.1673－3851(n).2020.06.016

Some properties of the complete p-elliptic integrals

QIU Songliang，WANG Jie，DING Zhishuan
(School of Science，Zhejiang Sci-Tech University，Hangzhou 310018，China)

Abstract：In this paper，the authors present some properties of the complete p-elliptic integrals $\overline{\mathcal{K}}_p(r)$ and $\overline{\mathcal{E}}_p(r)$ by showing the monotonicity properties of certain combinations defined in terms of $\overline{\mathcal{K}}_p(r)$，$\overline{\mathcal{E}}_p(r)$ and elementary functions，thus extending several results of monotonicity and concavity for the complete elliptic integrals to $\overline{\mathcal{K}}_p(r)$ and $\overline{\mathcal{E}}_p(r)$.

Key words：complete p-elliptic integrals；generalized trigonometric functions；monotonicity；inequality；convexity
CLC number：O174.6　　Document code：A　　ArticleID:1673-3851 (2020) 11-0852-06

完全 p-椭圆积分的一些性质

裘松良，王　婕，丁志栓
（浙江理工大学理学院，杭州 310018）

摘　要：通过研究由完全 p-椭圆积分 $\overline{\mathcal{K}}_p(r)$ 和 $\overline{\mathcal{E}}_p(r)$ 以及初等函数的适当组合的单调性等性质，揭示了完全 p-椭圆积分的一些性质，并将完全椭圆积分的一些有关单调性和凹凸性的结果推广到完全 p-椭圆积分 $\overline{\mathcal{K}}_p(r)$ 和 $\overline{\mathcal{E}}_p(r)$。

关键词：完全 p-椭圆积分；广义三角函数；单调性；不等式；凹凸性

1　Main Results

In the early 19th century，Gauss made outstanding contributions in the field of hypergeometric function. Since then，many mathematicians carried out in-depth research on this basis. For $a,b,c\in\mathbb{R}$ with $c\neq 0,-1,-2,\cdots$，the Gaussian hypergeometric function is defined by

$$F(a,b;c;z)={}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)n!}z^n(|z|<1) \tag{1}$$

where $(a,0)=1$ for $a\neq 0$，and $(a,n)=a(a+1)\cdots(a+n-1)$ for $n\in\mathbb{N}$ is the lifted factorial function，while $\mathbb{N}$ is the set of positive integers (cf. [1]). It is well known that many special functions and even some elementary functions are the particular or limiting cases of $F(a,b;c;z)$. For example，the well-known complete elliptic integrals of the first and second kinds are defined by

$$\begin{cases}\mathcal{K}=\mathcal{K}(r)=\pi F(1/2,1/2;1;r^2)/2,0<r<1,\\ \mathcal{K}'=\mathcal{K}'(r)=\mathcal{K}(\sqrt{1-r^2}),\\ \mathcal{K}(0)=\pi/2,\mathcal{K}(1)=\infty\end{cases}$$

and

Received date：2020－07－01　　Published Online：2020－09－03
Fund item：This research is supported by the NSF ofP. R. China (Grant No. 11771400)

Introduction to the first and corresponding author：QIU Songliang (1957－　)，male，Fuyang，Zhejiang，Professor；research interests：quasiconformal theory，special functions，Ramanuian's modular equations，etc. E-mail：sl_qiu@zstu.edu.cn

$$\begin{cases}\mathcal{E}(r)=\pi F(-1/2,1/2;1;r^2)/2, 0<r<1,\\ \mathcal{E}'=\mathcal{E}'(r)=\mathcal{E}(\sqrt{1-r^2}),\\ \mathcal{E}(0)=\pi/2, \mathcal{E}(1)=1\end{cases}$$

respectively (cf. [1]). Recently, the complete $p-$elliptic integrals were introduced (cf. [2]), which are related to the generalized p-trigonometric functions with the parameter $p\in(1,\infty)$.

For $x\in[0,1]$ and $p\in(1,\infty)$, the inverse p-sine function is defined by

$$\operatorname{arcsin}_p x=\int_0^x(1-t^p)^{-1/p}\mathrm{d}t=xF(1/p,1/p;1+1/p;x^p),$$

by which the constant π can be extended to following

$$\pi_p=2\operatorname{arcsin}_p 1=2\int_0^1\frac{\mathrm{d}t}{(1-t^p)^{1/p}}=\frac{2}{p}B\left(\frac{1}{p},1-\frac{1}{p}\right)=\frac{2\pi}{p\sin(\pi/p)},$$

where B is the classical beta function. The inverse of arcsin_p on $[0,\pi_p/2]$ is denoted by $\sin_p$, which is called the p-sine function and appears in the eigenfunction of Dirichlet problem of one-dimensional p-Laplacian (cf. [2－5]). If $p=2$, then $\sin_p x\equiv\sin x$ is the usual sine function, $\operatorname{arcsin}_p x\equiv\arcsin x$ and $\pi_p=\pi$.

Throughout this paper, we always assume that $p\in(1,\infty)$, and let $r'=(1-r^p)^{1/p}$ for $r\in[0,1]$. For $r\in(0,1)$, the complete p-elliptic integrals of the first and second kinds are defined as

$$\begin{cases}\mathcal{K}_p=\mathcal{K}_p(r)=\pi_p F(1/p,1-1/p;1;r^p)/2,\\ \mathcal{K}_p'=\mathcal{K}_p'(r)=\mathcal{K}_p(r'),\\ \mathcal{K}_p(0)=\pi_p/2, \mathcal{K}_p(1)=\infty\end{cases}$$

and

$$\begin{cases}\mathcal{E}_p=\mathcal{E}_p(r)=\pi_p F(-1/p,1/p;1;r^p)/2,\\ \mathcal{E}_p'=\mathcal{E}_p'(r)=\mathcal{E}_p(r'),\\ \mathcal{E}_p(0)=\pi_p/2, \mathcal{E}_p(1)=1\end{cases}$$

respectively, which have the following integral representations

$$\mathcal{K}_p(r)=\int_0^{\pi_p/2}\frac{\mathrm{d}t}{[1-(r\sin_p t)^p]^{1-1/p}}=\int_0^1\frac{\mathrm{d}t}{(1-t^p)^{1/p}(1-r^pt^p)^{1-1/p}}$$

and

$$\mathcal{K}_p(r)=\int_0^{\pi_p/2}[1-(r\sin_p t)^p]^{1/p}\mathrm{d}t=\int_0^1\left(\frac{1-r^pt^p}{1-t^p}\right)^{1/p}\mathrm{d}t$$

(cf. [2, 4]). Clearly, $\mathcal{K}_p=\mathcal{K}$ and $\mathcal{E}_p=\mathcal{E}$ in the case when $p=2$.

It is well known that the special functions above-mentioned have many applications in several fields of mathematics, as well as in physics and engineering (cf. [1-8]). Some properties of the complete p-elliptic integrals have been revealed. For instance, in 2016, Takeuchi proved the following generalized Legendre relation and derivative formulas (cf. [2])

$$\mathcal{K}_p(r)\mathcal{E}_p(r)+\mathcal{K}_p'(r)\mathcal{E}_p(r)-\mathcal{K}_p(r)\mathcal{K}_p'(r)=\frac{\pi_p}{2},$$

$$\frac{\mathrm{d}\mathcal{K}_p}{\mathrm{d}r}=\frac{\mathcal{E}_p(r)-r'^p\mathcal{K}_p(r)}{rr'^p},\frac{\mathrm{d}\mathcal{E}_p}{\mathrm{d}r}=\frac{\mathcal{E}_p(r)-\mathcal{K}_p(r)}{r}\tag{2}$$

Zhang recently proved the following double inequality (cf. [4])

$$\frac{\operatorname{arth}_p r}{r}<\mathcal{K}_p(r)<\frac{\pi_p}{2}\frac{\operatorname{arth}_p r}{r},$$

where arth_p is the inverse of generalized hyperbolic tangent function th_p and satisfies the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arth}_p x)=\frac{1}{1-x^p},\frac{\operatorname{arth}_p x}{x}=\sum_{n=0}^{\infty}\frac{x^{pn}}{pn+1}\tag{3}$$

In 2018, Huang et al revealed several properties of $\mathcal{K}_p$ and $\mathcal{E}_p$ (cf. [8]).

However, the study of the complete p-elliptic integrals is still in the initial phase, and many of their properties need to be revealed. In particular, one task of such kind of studies is to extend the known properties of the complete elliptic integrals to the complete p-elliptic integrals. Motivated by this, the main purpose of this paper is to present some properties of the complete p-elliptic integrals by showing the monotonicity properties of certain combinations defined in terms of the complete p-elliptic integrals and elementary functions, thus extending several well-known results for the complete elliptic integrals to the complete p-elliptic integrals. In particular, we shall extend [7, Lemma 3.32], [9, Theorem 2.5], [10, Lemma 5.2(7)], [11, Theorem 2.6, 2.9], [12, Theorem 3.1] and [13, Theorem 1.3] to the complete p-elliptic integrals. We now state our main results below.

Theorem 1 For $c\in\mathbb{R}$, define the functions f_1, f_2, f_3 and f_4 on $(0,1)$ by $f_1(r)=\overline{\mathcal{K}}_p(r)\overline{\mathcal{K}}'_p(r)$,

$$f_2(r)=\overline{\mathcal{K}}_p(r)^c+\overline{\mathcal{K}}'_p(r)^c,\ f_3(r)=(rr')^p[\overline{\mathcal{K}}_p(r)+\overline{\mathcal{K}}'_p(r)],$$

$$f_4(r)=(r'^p-r^p)\{[\overline{\mathcal{E}}'_p(r)-r^p\,\overline{\mathcal{K}}'_p(r)]-[\overline{\mathcal{E}}_p(r)-r'^p\,\overline{\mathcal{K}}_p(r)]\}.$$

a) f_1 is strictly decreasing on $(0,2^{-1/p}]$ and increasing on $[2^{-1/p},1)$, and log-convex on $(0,1)$. Moreover, $f_1(x)=f_1(y)$ if and only if $x=y$ or $x=y'$.

b) If $1-p^2/(p-1)\leqslant c<0\ (c>0)$, then f_2 is increasing (decreasing) on $(0,2^{-1/p}]$ and decreasing (increasing, respectively) on $[2^{-1/p},1)$. Moreover, if $1-p^2/(p-1)\leqslant c<0\ (c>0)$, then f_2 is concave (convex, respectively) on $(0,1)$. In particular, for $r\in(0,1)$ with $r\neq 2^{-1/p}$, if $1-p^2/(p-1)\leqslant c<0$, then

$$(\pi_p/2)^c<\overline{\mathcal{K}}_p(r)^c+\overline{\mathcal{K}}_p'(r)^c<2\,\overline{\mathcal{K}}_p(2^{-1/p})^c$$

and if $c>0$, then

$$\overline{\mathcal{K}}_p(r)^c+\overline{\mathcal{K}}_p'(r)^c>2\,\overline{\mathcal{K}}_p(2^{-1/p})^c.$$

c) f_3 is strictly increasing on $(0,2^{-1/p}]$ and decreasing on $[2^{-1/p},1)$, with $f_3(0^+)=f_3(1^-)=0$ and $f_3(2^{-1/p})=\overline{\mathcal{K}}_p(2^{-1/p})/2$.

d) f_4 is strictly decreasing on $(0,2^{-1/p}]$ and increasing on $[2^{-1/p},1)$ with $f_4(0^+)=f_4(1^-)=1$ and $f_4(2^{-1/p})=0$.

Theorem 2 a) The function $g_1(r)\equiv\overline{\mathcal{E}}_p(r)\overline{\mathcal{E}}'_p(r)$ is strictly increasing on $[0,2^{-1/p}]$ and decreasing on $[2^{-1/p},1]$. Moreover, $g_1(x)=g_1(y)$ if and only if $x=y$ or $x=y'$.

b) For $c\in\mathbb{R}$ and $r\in[0,1]$, let $g_2(r)=\overline{\mathcal{E}}_p(r)^c+\overline{\mathcal{E}}'_p(r)^c$. If $c\in(0,1]$ ($c\in(-\infty,0)$), then g_2 is strictly increasing (decreasing) on $[0,2^{-1/p}]$ and decreasing (increasing, respectively) on $[2^{-1/p},1]$.

c) The function $g_3(r)\equiv(rr')^p[\overline{\mathcal{E}}_p(r)+\overline{\mathcal{E}}'_p(r)]$ is strictly increasing on $[0,2^{-1/p}]$ and decreasing on $[2^{-1/p},1]$.

Theorem 3 For $c\in\mathbb{R}$, $n\in\mathbb{N}_0\equiv\mathbb{N}\cup\{0\}$ and $r\in(0,1)$, let

$$\alpha=\pi_p\,\frac{(p-1)^2+1}{2p(p-1)},\ \beta=\pi_p\,\frac{(1/p,n+1)(1-1/p,n+1)(pn+p+1)}{2[(n+1)!]^2},$$

$$P_n(r)=\frac{\pi_p}{2}\sum_{k=0}^{n}\frac{(1/p,k)(1-1/p,k)}{(k!)^2}r^{pk}\ \text{and}\ Q_n(r)=\sum_{k=0}^{n}\frac{r^{pk}}{pk+1},$$

and define the functions h_1, h_2, h_3 and h_4 on $(0,1)$ by

$$h_1(r)=r'^p\,\overline{\mathcal{K}}_p(r)^c/\overline{\mathcal{E}}_p(r),\ h_2(r)=\overline{\mathcal{K}}_p(r)-c\cdot\operatorname{arth}_p r-\pi_p/2,$$

$$h_3(r)=\frac{\pi_p^2/4-r'^p\,\overline{\mathcal{K}}_p(r)^2}{\overline{\mathcal{E}}_p(r)-r'^p\,\overline{\mathcal{K}}_p(r)}\ \text{and}\ h_4(r)=\frac{\overline{\mathcal{K}}_p(r)-P_n(r)}{(\operatorname{arth}_p r)/r-Q_n(r)}.$$

a) h_1 is decreasing if and only if $c\leqslant p+1$, with $h_1((0,1))=(0,(\pi_p/2)^{c-1})$. If $c>p+1$, then there exists a number $r_0\in(0,1)$ such that h_1 is increasing on $(0,r_0]$ and decreasing on $[r_0,1)$.

b) h_2 is strictly increasing (decreasing) on $(0,1)$ if only if $c\leqslant 0$ ($c\geqslant 1$, respectively), and if $0<c<1$, then there exists an $r_1\in(0,1)$ such that h_2 is strictly decreasing (increasing) on $(0,r_1]$ ($[r_1,1)$, respectively). Moreover, if $c\leqslant 0$, then h_2 is convex on $(0,1)$.

c) h_3 is strictly increasing from $(0,1)$ onto $(\alpha, \pi_p^2/4)$.

d) h_4 is strictly decreasing from $(0,1)$ onto $(1,\beta)$. In particular, for $r\in(0,1)$ and $n\in\mathbb{N}$,

$$P_n(r)+\frac{\mathrm{arth}_p r}{r}-Q_n(r)<\overline{\mathcal{K}}_p(r)<P_n(r)+\beta\left(\frac{\mathrm{arth}_p r}{r}-Q_n(r)\right) \tag{4}$$

2 Proof of Theorems

2.1 Proof of Theorem 1

a) Set $f_5(r)=\overline{\mathcal{K}}'_p(r)[\overline{\mathcal{E}}_p(r)-r'^p\overline{\mathcal{K}}_p(r)]$, which is strictly increasing from $(0,1)$ onto $(0,\pi_p/2)$ by [4, Lemma 3.4(1)&(6)]. By logarithmic differentiation and (2),

$$\frac{f_1'(r)}{f_1(r)}=\frac{r^{p-1}}{r'^{p-1+1/p}\cdot r'^{1-1/p}\overline{\mathcal{K}}_p(r)}\frac{\overline{\mathcal{E}}_p(r)-r'^p\overline{\mathcal{K}}_p(r)}{r^p}-$$

$$\frac{1}{r^{1/p}\cdot r^{1-1/p}\overline{\mathcal{K}}'_p(r)}\frac{\overline{\mathcal{E}}'_p(r)-r^p\overline{\mathcal{K}}'_p(r)}{r'^p}=\frac{f_5(r)-f_5(r')}{rr'^pf_1(r)} \tag{5}$$

It follows from (5) and [4, Lemma 3.4(1)&(6)] that the function $r\mapsto f_1'(r)/f_1(r)$ is strictly increasing on $(0,1)$, so that f_1' is negative (positive) on $(0,2^{-1/p}]$ ($[2^{-1/p},1)$, respectively) and has a unique zero $r=2^{-1/p}$ on $(0,1)$. Hence the piecewise monotonicity and log-convexity properties of f_1 follow.

Clearly, if $x=y$ or $x=y'$, then $f_1(x)=f_1(y)$. Conversely, suppose that $f_1(x)=f_1(y)$ with $x\neq y$. By the symmetry, we may assume that $x<y$. Then by part a), $0<x<2^{-1/p}<y<1$, $x'\in(2^{-1/p},1)$, $f_1(x')=f_1(x)=f_1(y)$, and hence $x'=y$.

b) Let $f_6(r)=f_7(r)-f_7(r')$, where

$$f_7(r)=r^p\overline{\mathcal{K}}'_p(r)^{1-c}\cdot r^{-p}[\overline{\mathcal{E}}_p(r)-r'^p\overline{\mathcal{K}}_p(r)].$$

It is clear that f_7 is strictly increasing on $(0,1)$ if $c\geqslant 1$. If $1-p^2/(p-1)\leqslant c<1$, then the function $r\mapsto r^p\overline{\mathcal{K}}'_p(r)^{1-c}$ is strictly increasing on $(0,1)$ by [4, Lemma 3.4(1)&(6)], and so is f_7. Hence if $1-p^2/(p-1)\leqslant c<\infty$, then f_7 is strictly increasing from $(0,1)$ onto $(0,(\pi_p/2)^{1-c})$ so that f_6 is strictly increasing from $(0,1)$ onto $(-(\pi_p/2)^{1-c},(\pi_p/2)^{1-c})$ with $f_6(2^{-1/p})=0$. Clearly, $1-p^2/(p-1)<0$. By differentiation and (2),

$$rr'^p[\overline{\mathcal{K}}_p(r)\overline{\mathcal{K}}'_p(r)]^{1-c}f_2'(r)=cf_6(r) \tag{6}$$

yielding the piecewise monotonicity property of f_2. It follows from (6) that

$$\frac{f_2'(r)}{c}=f_8(r)\equiv\frac{\overline{\mathcal{E}}_p(r)-r'^p\overline{\mathcal{K}}_p(r)}{r^p}\frac{r^{p-1}}{r'^p\overline{\mathcal{K}}_p(r)^{1-c}}-\frac{\overline{\mathcal{E}}'_p(r)-r^p\overline{\mathcal{K}}'_p(r)}{r'^p}\frac{1}{r\overline{\mathcal{K}}'_p(r)^{1-c}} \tag{7}$$

By investigating two cases when $c\geqslant 1$ and $c<1$, and applying [4, Lemma 3.4(1)&(6)], we see that the first (second) term in (7) is strictly increasing (decreasing) on $(0,1)$ if $1-p^2/(p-1)\leqslant c<\infty$ ($-1/(p-1)\leqslant c<\infty$, respectively). Clearly, $1-p^2/(p-1)<-1/(p-1)$. Hence if $-1/(p-1)\leqslant c<\infty$, then f_8 is strictly increasing on $(0,1)$. Consequently, it follows from (7) that if $-1/(p-1)\leqslant c<0$ ($c\in(0,\infty)$), then f_2 is concave (convex, respectively) on $(0,1)$.

c) The limiting values of f_3 are obvious. Differentiation gives

$$r^{1-p}f_3'(r)=f_9(r)\equiv\overline{\mathcal{E}}_p(r)-\overline{\mathcal{E}}'_p(r)+(p-1)[r'^p\overline{\mathcal{K}}_p(r)-r^p\overline{\mathcal{K}}'_p(r)]+p[r'^p\overline{\mathcal{K}}'_p(r)-r^p\overline{\mathcal{K}}_p(r)].$$

By [4, Lemma 3.4], f_9 is strictly decreasing on $(0,1)$ with $f_9(2^{-1/p})=0$. This yields part c).

d) The limiting values of f_4 are obvious. Since the function $f_{10}(r)\equiv\overline{\mathcal{E}}_p(r)-r'^p\overline{\mathcal{K}}_p(r)$ is strictly increasing from $(0,1)$ onto itself by [4, Lemma 3.4(1)], f_4 is a product of two positive (negative) and decreasing functions on $(0,2^{-1/p})$ ($(2^{-1/p},1)$, respectively). Hence the result for f_4 follows. □

2.2 Proof of Theorem 2

a) Let $g_4(r)=\overline{\mathcal{E}}'_p(r)\cdot r^{-p}[\overline{\mathcal{K}}_p(r)-\overline{\mathcal{E}}_p(r)]$, which is strictly increasing from $(0,1)$ onto $(\pi_p/(2p),$

∞) by [4, Lemma 3.4(4)]. By differentiation and (2),

$$r^{1-p}g_1'(r)=g_4(r')-g_4(r),$$

which is strictly decreasing from $(0,1)$ onto $(-\infty,\infty)$. Hence the piecewise monotonicity of g_1 follows. The proof of the remaining conclusion is similar to that of Theorem 1(a).

b) Let $g_5(r)=g_6(r')-g_6(r)$, where $g_6(r)=r^{-p}\,\overline{\mathcal{E}}_p(r)^{c-1}[\overline{\mathcal{K}}_p(r)-\overline{\mathcal{E}}_p(r)]$. Then by (2),

$$r^{1-p}g_2'(r)=cg_5(r) \tag{8}$$

If $-\infty<c\leqslant 1$, then g_6 is strictly increasing from $(0,1)$ onto $(\pi_p^c/(2^c p),\infty)$ by [4, Lemma 3.4(4)], and hence g_5 is strictly decreasing from $(0,1)$ onto $(-\infty,\infty)$ with $g_5(2^{-1/p})=0$. Consequently, the result for g_2 follows from (8).

c) Part c) follows from the piecewise monotonicity of $g_7(r)\equiv(rr')^p$ and part b) with $c=1$.□

2.3 Proof of Theorem 3

a) By differentiation and (2),

$$\frac{r\,\overline{\mathcal{K}}_p(r)^{1-c}\,\overline{\mathcal{E}}_p(r)}{\overline{\mathcal{E}}_p(r)-r'^p\,\overline{\mathcal{K}}_p(r)}h_1'(r)=c-\frac{h_5(r)}{h_6(r)}=c-h_7(r) \tag{9}$$

where $h_7(r)=h_5(r)/h_6(r)$,

$$h_5(r)=p-r'^p\,\frac{\overline{\mathcal{K}}_p(r)-\overline{\mathcal{E}}_p(r)}{r^p\,\overline{\mathcal{E}}_p(r)}\ \text{and}\ h_6(r)=\frac{\overline{\mathcal{E}}_p(r)-r'^p\,\overline{\mathcal{K}}_p(r)}{r^p\,\overline{\mathcal{K}}_p(r)}.$$

By [4, Lemma 3.4(3)&(5)], h_5 (h_6) is strictly increasing (decreasing) from $(0,1)$ onto $(p-1/p,p)$ ($(0,1-1/p)$, respectively), so that the function h_7 is strictly increasing from $(0,1)$ onto $(p+1,\infty)$. Hence by (9), h_1 is strictly decreasing on $(0,1)$ if and only if

$$c\leqslant\inf_{0<r<1}h_7(r)=h_7(0^+)=p+1.$$

The remaining conclusion in part a) is clear.

b) Differentiation gives

$$r'^p h_2'(r)=h_8(r)\equiv[\overline{\mathcal{E}}_p(r)-r'^p\,\overline{\mathcal{K}}_p(r)]/r-c \tag{10}$$

which is strictly increasing from $(0,1)$ onto $(-c,1-c)$ by [4, Lemma 3.4(1)]. Hence h_2 is strictly increasing (decreasing) on $(0,1)$ if and only if $c\leqslant 0$ ($c\geqslant 1$, respectively). In the case when $0<c<1$, the piecewise monotonicity of h_2 is clear. If $c\leqslant 0$, then h_8 is positive and strictly increasing on $(0,1)$, and hence by (10), we see that h_2 is convex on $(0,1)$.

c) Let $h_9(r)=(\pi_p/2)^2-r'^p\,\overline{\mathcal{K}}_p(r)^2$ and $h_{10}(r)=\overline{\mathcal{E}}_p(r)-r'^p\,\overline{\mathcal{K}}_p(r)$. Then $h_3(r)=h_9(r)/h_{10}(r)$, $h_9(0)=h_{10}(0)=0$ and

$$(p-1)h_9'(r)/h_{10}'(r)=\overline{\mathcal{K}}_p(r)\{p-2[\overline{\mathcal{E}}_p(r)-r'^p\,\overline{\mathcal{K}}_p(r)]/[r^p\,\overline{\mathcal{K}}_p(r)]\} \tag{11}$$

which is a product of two positive and increasing functions by [4, Lemma 3.4(3)]. Hence by [7, Theorem 1.25], h_3 is strictly increasing on $(0,1)$. Clearly, $h_3(1^-)=\pi_p^2/4$. By l'Hôpital's rule, [4, Lemma 3.4], and by (11), $h_3(0^+)=\pi_p[(p-1)^2+1]/[2p(p-1)]=\alpha$.

d) Let $h_{11}(r)=\overline{\mathcal{K}}_p(r)-P_n(r)$ and $h_{12}(r)=(\mathrm{arth}_p r)/r-Q_n(r)$. Then by (1) and (3),

$$h_{11}(r)=\frac{\pi_p}{2}\sum_{k=n+1}^{\infty}\frac{(1/p,k)(1-1/p,k)}{(k!)^2}r^{pk}=\frac{\pi_p}{2}r^{p(n+1)}\sum_0^{\infty}a_k r^{pk},$$

$$h_{12}(r)=\sum_{k=n+1}^{\infty}\frac{r^{pk}}{pk+1}=r^{p(n+1)}\sum_0^{\infty}b_k r^{pk}\ \text{and}\ h_4(r)=\frac{h_{11}(r)}{h_{12}(r)}=\frac{\pi_p}{2}\,\frac{\sum_0^{\infty}a_k r^{pk}}{\sum_0^{\infty}b_k r^{pk}},$$

where $a_k=(1/p,k+n+1)(1-1/p,k+n+1)[(k+n+1)!]^{-2}$ and $b_k=1/[p(k+n+1)+1]$. Putting $c_k=a_k/b_k$, we have

$$c_{k+1}/c_k=1-[p(k+n+1)]^{-2}<1,$$

which shows that the sequence $\{c_k\}$ is strictly decreasing. Hence the monotonicity property of h_4 follows from [14, Lemma 2.1].

Clearly, $h_4(0^+)=c_0\pi_p/2=\beta$. Applying l'Hôpital's rule, we obtain the limiting value $h_4(1^-)=1$. The double inequality (4) is obvious. □

References:

[1] Abramowitz M, Stegun I A. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables[M]. New York: Dover, 1966: 556-607.

[2] Takeuchi S. A new form of the generalized complete elliptic integrals[J]. Kodai Mathematical Journal, 2016, 39(1): 202-226.

[3] Lang J, Edmunds D. Trigonometric generalisations[M]//Eigenvalues, Embeddings and Generalized Trigonometric Functions[M]. Berlin: Springer, 2011: 33-48.

[4] Zhang X H. Monotonicity and functional inequalities for the complete p-elliptic integrals[J]. Journal of Mathematical Analysis and Applications, 2017, 453(2): 942-953.

[5] Lindqvist P. Note on a nonlinear eigenvalue problem[J]. Rocky Mountain Journal of Mathematics, 1993, 23(1): 281-288.

[6] Wang M K, Chu H H, Chu Y M. Precise bounds for the weighted Hölder mean of the complete p-elliptic integrals[J]. Journal of Mathematical Analysis and Applications, 2019, 480(2): 123388.

[7] Anderson G D, Vamanamurthy M K, Vuorinen M. Conformal Invariants, Inequalities, and Quasiconformal Maps[M]. New York: John Wiley and Sons, 1997: 48-50.

[8] Huang T R, Tan S Y, Ma X Y, et al. Monotonicity properties and bounds for the complete p-elliptic integrals[J]. Journal of Inequalities and Applications, 2018(1): 239-250.

[9] Anderson G D, Qiu S L, Vamanamurthy M K. Elliptic integral inequalities, with applications[J]. Constructive Approximation, 1998, 14(2): 195-207.

[10] Anderson G D, Qiu S L, Vamanamurthy M K, et al, Vuorinen M. Generalized elliptic integrals and modular equations [J]. Pacific Journal of Mathematics, 2000, 192(1): 1-37.

[11] Tu G Y, Qiu S L. Generalized Elliptic Integrals and the Properties of the Solutions of Ramanujan Modular Equations[D]. Hangzhou: Zhejiang Sci-Tech University, 2009: 17-19.

[12] 赵叶华. 广义椭圆积分的一些性质[D]. 杭州：杭州电子科技大学，2004：12-13.

[13] Ma X Y, Qiu S L, Tu G Y. Generalized Grötzsch ring function and generalized elliptic integrals[J]. Applied Mathematics: A Journal of Chinese Universities, 2016, 31(4): 458-468.

[14] Ponnusamy S, Vuorinen M. Asymptotic expansions and inequalities for hypergeometric function[J]. Mathematika, 1997, 44(2): 278-301.

(责任编辑:康　锋)