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Some properties of the (p, q)-Grötzsch ring function and (p, q)-Hübner function

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Abstract: The authors present some properties of the (p, q)-Grötzsch ring function $\tilde{\mu}_{p,q}(r)$ and the (p, q)-Hübner function $\mathcal{M}_{p,q}(r)$ which are defined by the complete (p, q)-elliptic integrals of the first kind, by showing the monotonicity and convexity properties of certain combinations defined in terms of $\tilde{\mu}_{p,q}(r)$, $\mathcal{M}_{p,q}(r)$ and elementary functions, thus extending some well-known results for the Grötzsch ring function and the Hübner function to $\tilde{\mu}_{p,q}(r)$ and $\mathcal{M}_{p,q}(r)$.

Key words: complete (p, q)-elliptic integral; (p, q)-Grötzsch ring function; (p, q)-Hübner function; monotonicity; convexity; inequality

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(p,q)-Grötzsch 环函数与(p,q)-Hübner 函数的一些性质

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摘　要： 对于 $r\in(0,1)$，通过揭示由第一类完全(p,q)-椭圆积分定义的(p,q)-Grötzsch 环函数 $\tilde{\mu}_{p,q}(r)$ 和(p,q)-Hübner 函数 $\mathcal{M}_{p,q}(r)$ 以及初等函数定义的一些组合的单调性、凹凸性，给出了 $\tilde{\mu}_{p,q}(r)$ 与 $\mathcal{M}_{p,q}(r)$ 的一些性质，从而将 Grötzsch 环函数和 Hübner 函数的一些已知结果推广到 $\tilde{\mu}_{p,q}(r)$ 和 $\mathcal{M}_{p,q}(r)$。

关键词： 完全(p,q)-椭圆积分；(p,q)-Grötzsch 环函数；(p,q)-Hübner 函数；单调性；凹凸性；不等式

1　Main Results

In the 18th century, the mathematician Euler studied the analytical properties of hypergeometric functions deeply and gave the integral expression of hypergeometric functions. At the same time, Gauss promoted the rapid development of hypergeometric functions, which aroused more and more mathematicians' interest in hypergeometric functions. For real a, b and c ($c\neq 0,-1,-2,\cdots$), the Gaussian hypergeometric function is defined by

$$F(a,b;c;x)={}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)n!}x^n(|x|<1),$$

where $(a,0)=1$ for $a\neq 0$, and $(a,n)=a(a+1)\cdots(a+n-1)$ for $n\in\mathbb{N}$ is the shifted factorial function, while $\mathbb{N}$ is the set of positive integers (cf. [1-2]). It is well known that the hypergeometric functions have many applications in mathematics, physics and engineering, and many other special functions and even elementary functions are the particular or limiting cases of $F(a,b;c;x)$. For instance, the complete

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(p,q)-elliptic integrals introduced recently are particular cases of $F(a,b;c;x)$ and related to the generalized (p,q)-trigonometric functions.

For $p,\ q\in(1,\infty)$ and $x\in[0,1]$, the generalized (p,q)-arcsine function is defined by

$$\arcsin_{p,q}x=\int_0^x (1-t^q)^{-1/p}\,\mathrm{d}t,$$

by which the constant π can be extended to the following

$$\pi_{p,q}=2\arcsin_{p,q}1=2\int_0^1 (1-t^q)^{-1/p}\,\mathrm{d}t=\frac{2}{q}B(1-1/p,1/q) \tag{1}$$

where B is the beta function. The inverse of $\arcsin_{p,q}x$ on $[0,\pi_{p,q}/2]$ is called the generalized (p,q)-sine function denoted by $\sin_{p,q}$, which is an eigenfunction of the Dirichlet problem for the (p,q)-Laplacian and can be extended to a $2\pi_{p,q}$-periodic function on $\mathbb{R}$ (See [3-4]).

For $p,\ q\in(1,\infty)$ and $r\in(0,1)$, the complete (p,q)-elliptic integrals of the first and the second kinds are defined by (cf. [5])

$$\begin{cases}\mathcal{K}_{p,q}=\mathcal{K}_{p,q}(r)=\pi_{p,q}F(1-1/p,1/q;1-1/p+1/q;r^q)/2,\\ \mathcal{K}'_{p,q}=\mathcal{K}'_{p,q}(r)=\mathcal{K}_{p,q}(r'),\\ \mathcal{K}_{p,q}(0)=\pi_{p,q}/2,\mathcal{K}_{p,q}(1)=\infty\end{cases} \tag{2}$$

and

$$\begin{cases}\mathcal{E}_{p,q}=\mathcal{E}_{p,q}(r)=\pi_{p,q}F(-1/p,1/q;1-1/p+1/q;r^q)/2,\\ \mathcal{E}'_{p,q}=\mathcal{E}'_{p,q}(r)=\mathcal{E}_{p,q}(r'),\\ \mathcal{E}_{p,q}(0)=\pi_{p,q}/2,\mathcal{E}_{p,q}(1)=1\end{cases} \tag{3}$$

respectively. Here and hereafter, $r'=(1-r^q)^{1/q}$ for $r\in[0,1]$. Clearly, $\mathcal{K}_{2,2}=\mathcal{K}$ and $\mathcal{K}'_{2,2}=\mathcal{K}'$ $\mathcal{E}_{2,2}=\mathcal{E}$ and $\mathcal{E}'_{2,2}=\mathcal{E}'$) are thecomplete elliptic integrals of the first(second, respectively) kind.

In the sequel, we always assume that $p,q\in(1,\infty)$, and let $a=1-1/p$, $b=1/q$, $c=a+b$, $d=\dfrac{p-2+\sqrt{5p^2-8p+4}}{2(p-1)}$, $\beta=bR(a,b)$ and $\eta=b\left(1-2\dfrac{ab}{c}\right)=\dfrac{1}{q}\left[1-\dfrac{2(p-1)}{q(p-1)+p}\right]$. where $R(a,b)=-2\gamma-\psi(a)-\psi(b)$ is the Ramanujan R-function (or the Ramanujan constant), $\gamma=0.577215\cdots$ is the Euler-Mascheroni constant, and ψ is the classical psi function.

Let $m(r)=2(1-r^2)\mathcal{K}(r)\mathcal{K}'(r)/\pi$ and $m_{p,q}(r)=2r'^q\mathcal{K}_{p,q}(r)\mathcal{K}'_{p,q}(r)/\pi_{p,q}$. It is well known that the Grötzsch ring function $\mu(r)=\pi\mathcal{K}'(r)/[2\mathcal{K}(r)]$ and the Hübner function $\mathcal{M}(r)=m(r)+\log r$ are indispensable in the studies of the theories of quasiconformal mapings and Ramanujan's modular equations (cf. [2,6-11]). We now extend $\mu(r)$ and $\mathcal{M}(r)$ to the following

$$\tilde{\mu}_{p,q}(r)=\frac{\pi_{p,q}}{2}\frac{\mathcal{K}'_{p,q}(r)}{\mathcal{K}_{p,q}(r)},\mathcal{M}_{p,q}(r)=m_{p,q}(r)+\log r \tag{4}$$

For $r\in(0,1)$, and call them the (p,q)-Grötzsch ring function and the (p,q)-Hübner function, respectively. These functions can play a role in the study of a kind of modular equations.

It is natural to ask whether the known results for $\mu(r)$ and $\mathcal{M}(r)$ can be extended to $\tilde{\mu}_{p,q}(r)$ and $\mathcal{M}_{p,q}(r)$. This is a problem worth studying. Motivated by this, the purpose of this paper is to study the properties of the functions $\tilde{\mu}_{p,q}(r)$ and $\mathcal{M}_{p,q}(r)$, and extend several well-known results for $\mu(r)$ and $\mathcal{M}(r)$ to $\tilde{\mu}_{p,q}(r)$ and $\mathcal{M}_{p,q}(r)$, by showing the monotonicity and convexity properties of certain combinations defined in terms of $\tilde{\mu}_{p,q}(r)$, $\mathcal{M}_{p,q}(r)$ and elementary functions. We now state some of our main results.

Theorem 1 a) The function $\mathcal{M}_{p,q}$ is strictly decreasing from $(0,1)$ onto $(0,\beta)$. Moreover, if $q\geqslant d$, then $\mathcal{M}_{p,q}$ is concave on $(0,1)$.

b) The function $f_1(r)\equiv m_{p,q}(r)/\log(1/r)$ is strictly increasing from $(0,1)$ onto $(1,\infty)$.

c) For each $t\in(0,1)$, the function $f_2(r)\equiv m_{p,q}(rt)-m_{p,q}(r)$ is strictly increasing from $(0,1)$ onto $(-\log t, m_{p,q}(t))$. In particular, for all $r,t\in(0,1)$,

$$\max\{m_{p,q}(r)-\log t, m_{p,q}(t)-\log r\}<m_{p,q}(rt)<m_{p,q}(r)+m_{p,q}(t) \tag{5}$$

$$\max\{\mathcal{M}_{p,q}(r),\mathcal{M}_{p,q}(t)\}<\mathcal{M}_{p,q}(rt)<\mathcal{M}_{p,q}(r)+\mathcal{M}_{p,q}(t) \tag{6}$$

d) For each $\alpha\in(0,\infty)$, the function $f_3(r)\equiv\alpha m_{p,q}(r)-m_{p,q}(r^\alpha)$ is strictly increasing (decreasing) from $(0,1)$ onto $((\alpha-1)\beta,0)$ ($(0,(\alpha-1)\beta)$) if and only if $0<\alpha<1$ ($\alpha>1$, respectively). Moreover, if $\alpha\in(0,1)$, then for $r\in(0,1)$,

$$\alpha m_{p,q}(r)<m_{p,q}(r^\alpha)<\alpha m_{p,q}(r)+\beta(1-\alpha) \tag{7}$$

and if $\alpha\in(1,\infty)$, then each inequality in (7) is reversed.

Theorem 2 a) The function $g_1(r)\equiv\tilde{\mu}_{p,q}(r)+\log r$ is strictly decreasing and concave from $(0,1)$ onto $(0,\beta)$, and $g_2(r)\equiv g_1(r)/r'^q$ is strictly increasing from $(0,1)$ onto (β,∞).

b) $g_3(r)\equiv\tilde{\mu}_{p,q}(1/r)/\log r$ is strictly decreasing and convex from $(1,\infty)$ onto itself.

c) $g_4(r)\equiv g_1(r)-\log r'$ is strictly increasing and convex from $(0,1)$ onto (β,∞).

d) $g_5(r)\equiv\tilde{\mu}_{p,q}(r)\log(1/r')$ is strictly increasing from $(0,1)$ onto $(0,\pi_{p,q}{}^2/4)$.

e) $g_6(r)=[\beta-g_1(r)]/r^q$ is strictly increasing from $(0,1)$ onto (η,β). In particular,

$$\beta r'^q<\tilde{\mu}_{p,q}(r)+\log r<\beta r'^q+(\beta-\eta)r^q, r\in(0,1) \tag{8}$$

Theorem 3 For each $t\in(0,1)$, define the functions h_1,h_2 and h_3 on $(0,1)$ by $h_1(r)=\tilde{\mu}_{p,q}(rt)-\tilde{\mu}_{p,q}(r)$, $h_2(r)=\tilde{\mu}_{p,q}(rt)/\tilde{\mu}_{p,q}(r)$ and $h_3(r)=2\tilde{\mu}_{p,q}(\sqrt{rt})-\tilde{\mu}_{p,q}(r)$.

a) The functions h_1 and h_2 are both strictly increasing on $(0,1)$, with ranges $(-\log t,\tilde{\mu}_{p,q}(t))$ and $(1,\infty)$, respectively. In particular, for all $r,t\in(0,1)$,

$$\tilde{\mu}_{p,q}(r)+\log(1/t)<\tilde{\mu}_{p,q}(rt)<\tilde{\mu}_{p,q}(r)+\tilde{\mu}_{p,q}(t) \tag{9}$$

b) The function h_3 is strictly decreasing from $(0,t]$ onto $[\tilde{\mu}_{p,q}(t),\beta-\log t)$, and increasing from $[t,1)$ onto $[\tilde{\mu}_{p,q}(t),2\tilde{\mu}_{p,q}(\sqrt{t}))$. In particular, for all $r,t\in(0,1)$,

$$\tilde{\mu}_{p,q}(r)+\tilde{\mu}_{p,q}(t)\leqslant 2\tilde{\mu}_{p,q}(\sqrt{rt})<\tilde{\mu}_{p,q}(r)+\max\{\beta-\log t, 2\tilde{\mu}_{p,q}(\sqrt{t})\} \tag{10}$$

with equality if and only if $r=t$.

2 Preliminaries

First, let us recall the following generalized Legendre relation and derivative formulas[5]

$$\mathcal{K}'_{p,q}\mathcal{E}_{p,q}+\mathcal{K}_{p,q}\mathcal{E}'_{p,q}-\mathcal{K}_{p,q}\mathcal{K}'_{p,q}=\frac{\pi_{p,q}}{2} \tag{11}$$

$$\frac{\mathrm{d}\mathcal{K}_{p,q}}{\mathrm{d}r}=\frac{\mathcal{E}_{p,q}-r'^q\mathcal{K}_{p,q}}{rr'^q},\ \frac{\mathrm{d}\mathcal{K}'_{p,q}}{\mathrm{d}r}=-\frac{\mathcal{E}'_{p,q}-r^q\mathcal{K}'_{p,q}}{rr'^q} \tag{12}$$

$$\frac{\mathrm{d}\mathcal{E}_{p,q}}{\mathrm{d}r}=-q\frac{\mathcal{K}_{p,q}-\mathcal{E}_{p,q}}{pr},\ \frac{\mathrm{d}\mathcal{E}'_{p,q}}{\mathrm{d}r}=qr^{q-1}\frac{\mathcal{K}'_{p,q}-\mathcal{E}'_{p,q}}{pr'^q} \tag{13}$$

and the well-known asymptotic formula

$$\mathcal{K}_{p,q}(r)=\log\frac{e^\beta}{r'}+O((1-r^q)\log(1-r^q)) \tag{14}$$

as $r\to1$. Applying (11)-(13), one can easily prove the following lemma.

Lemma 1 For $r\in(0,1)$,

$$\frac{\mathrm{d}\tilde{\mu}_{p,q}(r)}{\mathrm{d}r}=-\frac{\pi_{p,q}^2}{4rr'^q\mathcal{K}_{p,q}^2}=-\frac{1}{rr'^qF(a,b;c;r^q)^2} \tag{15}$$

$$\begin{aligned}\frac{\mathrm{d}m_{p,q}(r)}{\mathrm{d}r}&=\frac{1}{\pi_{p,q}r}[\pi_{p,q}-4\mathcal{K}_{p,q}\mathcal{E}'_{p,q}-2(q-2)r^q\mathcal{K}_{p,q}\mathcal{K}'_{p,q}]\\&=-\frac{1}{r}-\frac{2}{\pi_{p,q}}r^{q-1}\mathcal{K}_{p,q}\mathcal{K}'_{p,q}\left(q-2\frac{\mathcal{E}_{p,q}-r'^q\mathcal{K}_{p,q}}{r^q\mathcal{K}_{p,q}}\right)\end{aligned} \tag{16}$$

Next, we present several properties of $\mathcal{K}_{p,q}$ and $\mathcal{E}_{p,q}$, which are the analogues of the related properties of $\mathcal{K}$ and $\mathcal{E}$ (cf. [8, Lemma 5.2(8), (14) & 5.4(3)—(4)]) for $\mathcal{K}_{p,q}$ and $\mathcal{E}_{p,q}$.

Lemma 2 Let $\delta = a\pi_{p,q}/(2c)$ and $\rho = \pi_{p,q}^2(1-2ab/c)/4$.

a) $H_1(r) \equiv (\mathcal{K}_{p,q} - \pi_{p,q}/2)/\log(1/r')$ is strictly increasing from $(0,1)$ onto $(\delta,1)$.

b) $H_2(r) \equiv (\pi_{p,q}^2/4 - r'^q \mathcal{K}_{p,q}^2)/r^q$ is strictly increasing from $(0,1)$ onto $(\rho, {\pi_{p,q}}^2/4)$.

c) Let $H_3(r) = (2/q)\mathcal{E}_{p,q} + (1-2/q) r'^q \mathcal{K}_{p,q}$. Then H_3 is strictly increasing from $(0,1)$ onto $(\pi_{p,q}/2, 2b)$ if and only if $1 < q \leqslant 2(1-1/p)$, and H_3 is strictly decreasing from $(0,1)$ onto $(2b, \pi_{p,q}/2)$ if and only if $q \in S_q \equiv [2,\infty) \cup \{q \mid q \in (1,2), q \geqslant 2-1/(p-1)$. Moreover, if $q \in (1,2]$ with $q \geqslant 2-1/(p-1)$, then H_3 is concave on $(0,1)$.

d) If $1 < q \leqslant 2(1-1/p)$ $(q \in S_q)$, then $H_4(r) \equiv r^{q/2}\mathcal{K}_{p,q}/\mathrm{arth}\, r^{q/2}$ is strictly increasing (decreasing) from $(0,1)$ onto $(\pi_{p,q}/2, 2b)$ ($(2b, \pi_{p,q}/2)$, respectively).

Proof a) Let $H_5(r) = \mathcal{K}_{p,q} - \pi_{p,q}/2$ and $H_6(r) = \log(1/r')$ for $r \in (0,1)$. Then $H_5(0) = H_6(0) = 0$, $H_1(r) = H_5(r)/H_6(r)$, and by (12),

$$H'_5(r)/H'_6(r) = (\mathcal{E}_{p,q} - r'^q \mathcal{K}_{p,q})/r^q,$$

which is strictly increasing on $(0,1)$ by [6, Theorem 1(a)]. Hence the monotonicity of H_1 follows from [2, Theorem 1.25]. By l'Hôpital's rule, we obtain the values $H_1(0^+) = \delta$ and $H_1(1^-) = 1$.

b) Let $H_7(r) = {\pi_{p,q}}^2/4 - r'^q \mathcal{K}_{p,q}^2$, $H_8(r) = r^q$, $H_9(r) = (\mathcal{E}_{p,q} - r'^q \mathcal{K}_{p,q})/(r^q \mathcal{K}_{p,q})$ and $H_{10}(r) = 1 - 2bH_9(r)$. Then $H_7(0) = H_8(0) = 0$, $H_2(r) = H_7(r)/H_8(r)$ and

$$H'_7(r)/H'_8(r) = \mathcal{K}_{p,q}^2 H_{10}(r) \tag{17}$$

By [6, Theorem 1(c)], H_9 is strictly decreasing from $(0,1)$ onto $(0, a/c)$, so that H_{10} is strictly increasing from $(0,1)$ onto $(1-2ab/c, 1)$. It is easy to verify that $2ab/c < 1$. Hence it follows from (17) and [2, Theorem 1.25] that H_2 is strictly increasing on $(0,1)$. By l'Hôpital's rule and (17), $H_2(0^+) = \rho$. Clearly, $H_2(1^-) = {\pi_{p,q}}^2/4$.

c) The limiting values of H_3 are clear. Set $H_{11}(r) = H_{14}(r)/r$, $H_{12}(r) = r^q \mathcal{K}_{p,q}/H_{14}(r)$ and $H_{13}(r) = 2/p + (1-2/q)[1+(q-1)H_{12}(r)]$, where $H_{14}(r) = \mathcal{K}_{p,q} - \mathcal{E}_{p,q}$. Then H_{11} (H_{12}) is strictly increasing (decreasing) from $(0,1)$ onto $(0,\infty)$ ($(1, c/b)$, respectively) by [6, Theorem 1(d)], $H_{13}(0^+) = q[q(p-1)-2p+3]/p$, and $H_{13}(1^-) = q - 2(1-1/p)$. By (12)—(13), we have

$$H'_3(r) = -H_{11}(r)H_{13}(r) \tag{18}$$

$$H_{13}(0^+) \leqslant 0 \Leftrightarrow q \leqslant 2 - 1/(p-1) \tag{19}$$

$$H_{13}(0^+) \geqslant 0 \Leftrightarrow q \geqslant 2 - 1/(p-1) \tag{20}$$

$$H_{13}(1^-) \geqslant 0 \Leftrightarrow q \geqslant 2(1 - 1/p) \tag{21}$$

$$H_{13}(1^-) \leqslant 0 \Leftrightarrow q \leqslant 2(1 - 1/p) \tag{22}$$

If $q \geqslant 2$, then $H_{13}(r) > 0$ for $r \in (0,1)$ so that H_3 is strictly decreasing on $(0,1)$ by (18).

If $q \in (1,2)$, then H_{13} is strictly increasing on $(0,1)$, and hence by (18)—(22), $H'_3(r) > 0$ $(H'_3(r) < 0)$ for $r \in (0,1)$ if and only if $H_{13}(1^-) \leqslant 0$ $(H_{13}(0^+) \geqslant 0)$, namely, $q \leqslant 2(1-1/p)$ ($q \geqslant 2 - 1/(p-1)$, respectively). Hence the result for H_3 follows.

d) Let $H_{15}(r) = r^{q/2}\mathcal{K}_{p,q}$ and $H_{16}(r) = \mathrm{arth}\, r^{q/2}$. Then $H_{15}(0) = H_{16}(0) = 0$, $H_4(r) = H_{15}(r)/H_{16}(r)$, and $H'_{15}(r)/H'_{16}(r) = H_3(r)$. Hence the assertion on the monotonicity properties of H_4 follows from partc). Clearly, $H_4(0^+) = \pi_{p,q}/2$ and $H_4(1^-) = H_3(1^-) = 2b$. □

3 Proof of Main Results

3.1 Proof of Theorem 1

a) Clearly, $\mathcal{M}_{p,q}(1^-) = 0$. Applying (14), we obtain the value $\mathcal{K}_{p,q}(0^+) = \beta$. Let H_{10} be as in the proof of Lemma 2(b). Then by (16),

$$-\pi_{p,q}\mathcal{M}'_{p,q}(r)=f_4(r)\equiv 2q(r^{q-1}\mathcal{K}'_{p,q})\cdot\mathcal{K}_{p,q}\cdot H_{10}(r),$$

which is positive for $r\in(0,1)$. Hence the monotonicity of $\mathcal{M}_{p,q}$ follows.

Clearly, $q-1\geqslant a/c$ if and only if $q\geqslant d$. Hence by [6, Theorem 2(a)], if $q\geqslant d$, then the function $r\mapsto r^{q-1}\mathcal{K}'_{p,q}(r)$ is strictly increasing from $(0,1)$ onto $(0,\pi_{p,q}/2)$, so that f_4 is a product of three positive and strictly increasing functions. This yields the concavity of $\mathcal{M}_{p,q}$.

b) Let $f_5(r)=m_{p,q}(r)$, $f_6(r)=\log(1/r)$ and $f_7(r)=r^q\mathcal{K}'_{p,q}\mathcal{K}_{p,q}H_{10}(r)$. Then f_7 is strictly increasing on $(0,1)$ by [6, theorem 2(a)], $f_1(r)=f_5(r)/f_6(r)$, $f_5(1)=f_6(1)=0$ and

$$f'_5(r)/f'_6(r)=f_8(r)\equiv -rm'_{p,q}(r)=1+2qf_7(r)/\pi_{p,q},$$

which is strictly increasing on $(0,1)$ and hence so is f_1 by [2, Theorem 1.25]. Applying l'Hôpital's rule, we obtain the limiting values of f_1.

c) Clearly, $f_2(1^-)=m_{p,q}(t)$ and $f_2(0^+)=-\log t$. Let $x=rt$. Then $0<x<r$, and

$$rf'_2(r)=r[tm'_{p,q}(x)-m'_{p,q}(r)]=f_8(r)-f_8(x).$$

Hence the result for f_2 follows from that of f_8. The double inequalities (5) and (6) are clear.

d) Clearly, $f_3(1^-)=0$. By part a), $f_3(0^+)=\beta(\alpha-1)$. Put $y=r^\alpha$. Then by differentiation,

$$rf'_3(r)=\alpha[rm'_{p,q}(r)-ym'_{p,q}(y)]=\alpha[f_8(y)-f_8(r)].$$

Hence by the monotonicity of f_8, $f'_3(r)>0$ ($f'_3(r)<0$) if and only if $y>r$ ($y<r$, respectively). This yields the monotonicity properties of f_3. The remaining conclusions are clear. □

3.2 Proof of Theorem 2

a) Clearly, $g_1(1^-)=0$. By (14), one can obtain the limiting value $g_1(0^+)=\beta$. Let H_2 be as in Lemma 2. Then by differentiation and (15)

$$-g'_1(r)=r^{q-1}\cdot H_2(r)\cdot(r'^q\mathcal{K}^2_{p,q})^{-1}\tag{23}$$

which is a product of three positive and strictly increasing on $(0,1)$ by Lemma 2(b) and [6, Theorem 2(a)]. Hence the monotonicity and concavity properties of g_1 follow.

Clearly, $g_2(0^+)=g_1(0^+)=\beta$. By l'Hôpital's rule, we obtain the value $g_2(1^-)=\infty$. By(23),

$$g'_1(r)\,[\mathrm{d}(r'^q)/\mathrm{d}r]^{-1}=bH_2(r)\cdot(r'^q\mathcal{K}^2_{p,q})^{-1},$$

which is a product of two positive and strictly increasing on $(0,1)$ by [6, Theorem 2(a)]. Hence the result for g_2 follows from [2, Theorem 1.25].

b) Applying l'Hôpital's rule and (14), one can obtain the limiting values of g_3.

It is easy to see that the function $x\mapsto x/\log(1/x)$ is strictly increasing on $(0,1)$. Let f_1 be as in Theorem 1, $x=1/r$, and $g_7(x)=\tilde{\mu}_{p,q}(x)/\log(1/x)$. Then $g_3(r)=g_7(x)$, and by (15),

$$-g'_3(r)=-g'_7(x)\frac{\mathrm{d}x}{\mathrm{d}r}=\frac{x}{\log(1/x)}\cdot\frac{{\pi_{p,q}}^2}{4x'^q\mathcal{K}_{p,q}(x)^2}\cdot[f_1(x)-1],$$

which is a product of three positive and strictly increasing functions of $x\in(0,1)$ by [6, Theorem 2(a)] and Theorem 1 b). Hence the result for g_3 follows.

c) The limiting values of g_4 follow from part a). It is easy to verify that the function $r\mapsto[\log(1/r')]/r$ is strictly increasing from $(0,1)$ onto $(0,\infty)$. Let H_1 be as in Lemma 2. Then by differentiation and (15),

$$g'_4(r)=\frac{\mathcal{K}_{p,q}+\pi_{p,q}/2}{r'^q\mathcal{K}^2_{p,q}}\cdot\frac{\log(1/r')}{r}\cdot H_1(r),$$

which is a product of three positive and strictly increasing functions on $(0,1)$ by Lemma 2 and [6, Theorem 2(a)]. Hence the result for g_4 follows.

d) Let $g_8(r)=\log(1/r')$ and $g_9(r)=1/\tilde{\mu}_{p,q}(r)$. Then $g_5(r)=g_8(r)/g_9(r)$, $g_8(0)=g_9(0)=0$ and $g'_8(r)/g'_9(r)=r^q\mathcal{K}'^2_{p,q}$, which is strictly increasing on$(0,1)$ by [6, Theorem2(a)]and hence so is g_5 by [2, Theorem1.25]. By l'Hôpital's rule, we obtain the limiting values of g_5.

e) Clearly, $g_6(1^-)=\beta$. By l'Hôpital's rule, we obtain $g_6(0^+)=\eta$. Let H_2 be as in Lemma 2, $g_{10}(r)=\beta-g_1(r)$ and $g_{11}(r)=r^q$. Then ${}_6(r)=g_{10}(r)/g_{11}(r)$, $g_{10}(0^+)=g_{11}(0)=0$ and

$$\frac{g'_{10}(r)}{g'_{11}(r)}=b\,\frac{H_2(r)}{r'^q\,\mathcal{K}^2_{p,q}},$$

which is strictly increasing on $(0,1)$ by [6, Theorem 2(a)] and Lemma 2(b). Hence the result for g_6 follows from [2, Theorem 1.25]. The double inequality (8) is clear. □

3.3 Proof of Theorem 3

a) Clearly, $h_1(1^-)=\tilde{\mu}_{p,q}(t)$. By Theorem 2 (a), $h_1(0^+)=-\log t$.

Let $x=rt$ and $h_4(r)=1/(r'^q\mathcal{K}^2_{p,q})$. Then $0<x<r<1$, and by differentiation,

$$4rh'_1(r)=\pi_{p,q}{}^2[h_4(r)-h_4(x)],$$

which is positive by [6, Theorem 2(a)]. This yields the result for h_1. The double inequality (9) is clear.

It is clear that $h_2(1^-)=\infty$ and $h_2(0^+)=1$. Let $h_5(r)=1/(r'^q\mathcal{K}_{p,q}\mathcal{K}'_{p,q})$. Then by (15),

$$2rh'_2(r)=\pi_{p,q}h_2(r)[h_5(r)-h_5(x)],$$

which is positive by [6, Theorem 2(a)]. Hence the result of h_2 follows.

b) It is clear that $h_3(t)=\tilde{\mu}_{p,q}(t)$, $h_3(1^-)=2\tilde{\mu}_{p,q}(\sqrt{t})$ and $h_3(0^+)=\beta-\log t$. Put $y=\sqrt{rt}$. Then $y>r$ ($y<r$) if and only if $r\in(0,t)$ ($r\in(t,1)$, respectively). By differentiation and (15),

$$4rh'_3(r)=\pi_{p,q}{}^2[h_4(r)-h_4(y)].$$

Hence by [6, Theorem 2(a)], $h'_3(r)<0$ ($h'_3(r)>0$) if and only $y>r$ ($y<r$, respectively). This yields the result for h_3. The double inequality (10) and its equality case are clear. □

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