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A double inequality for the ratio of complete elliptic integrals of the first kind

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Abstract: In this paper, the authors present a double inequality for the ratio $\mathscr{K}(r)/\mathscr{K}(\sqrt{r})$ of complete elliptic integrals of the first kind, in which the upper bound is much better than those known to us, while the proof of the lower bound is much simpler than that recently given by Alzer and Richards.

Key words: complete elliptic integral; monotonicity; lower and upper bound; inequality

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0 Introduction

Throughout this paper, we always let $r'=\sqrt{1-r^2}$ for each $r\in[0,1]$. As usual, the complete elliptic integrals of the first and second kinds are defined as

$$\mathscr{K}=\mathscr{K}(r)=\int_0^{\pi/2}\frac{1}{\sqrt{1-r^2\sin^2 t}}\mathrm{d}t,\mathscr{K}'=\mathscr{K}'(r)=\mathscr{K}(r'),0<r<1 \tag{1}$$

and

$$\mathscr{E}=\mathscr{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2 t}\,\mathrm{d}t,\mathscr{E}=\mathscr{E}'(r)=\mathscr{E}(r'),0<r<1 \tag{2}$$

respectively, with $\mathscr{K}(0)=\mathscr{E}(0)=\pi/2$, $\mathscr{K}(1^-)=\infty$ and $\mathscr{E}(1)=1$. The basic properties of $\mathscr{K}$ and $\mathscr{E}$ are collected, for instance, in [1]589-592 and [2-4]. It is well known that these special functions have many important applications in mathematics, physics and engineering. In particular, they play an important role in quasiconformal theory.

During the past decades, many authors have obtained various properties for $\mathscr{K}$ and $\mathscr{E}$, including functional inequalities (cf. [5-14]). In [5, Theorem 3.11], for example, the following double inequality was obtained

$$\frac{1}{\sqrt[4]{1+r}}<\frac{\mathscr{K}(r)}{\mathscr{K}(\sqrt{r})}<\frac{\min\{\sqrt[4]{2},1/\sqrt{r'}\}}{\sqrt[4]{1+r}},r\in[0,1] \tag{3}$$

while it was proved in [6, Theorem 1.1] that the function $r\mapsto\sqrt[4]{1+r}\,\mathscr{K}(r)/\mathscr{K}(\sqrt{r})$ is strictly increasing from $[0,1)$ onto $[1,\sqrt[4]{2})$. Recently, Alzer and Richards proved in [7] that the double inequality

$$\frac{4}{4+r}<\frac{\mathscr{K}(r)}{\mathscr{K}(\sqrt{r})}<1 \tag{4}$$

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holds for $r\in(0,1)$. However, the second inequality in (4) is sharp only in the sense that $\lim_{r\to 1}\mathcal{K}(r)/\mathcal{K}(\sqrt{r})=1$, and the proof of the first inequality in (4) given in [7] is quite complicated.

In this paper, motivated by (3) and (4), the authors intend to refine the upper bound in (4), and simplify the proof of the first inequality in (4) given in [7] to a great extent. In addition, we shall show some monotonicity properties of certain combinations in terms of $\mathcal{K}(r)$, $\mathcal{K}(\sqrt{r})$ and elementary functions. Our main results are stated in the following theorem.

Theorem 1 a) The function $f(r)\equiv(4+r)\mathcal{K}(r)/\mathcal{K}(\sqrt{r})$ is strictly increasing from $[0,1)$ onto $[4,5)$.

b) The function $g(r)\equiv(4+r)\mathcal{K}(r)-4\mathcal{K}(\sqrt{r})$ is strictly increasing from $[0,1)$ onto $[0,\infty)$.

c) For all $r\in[0,1)$,

$$\frac{4}{4+r}\leqslant\frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})}\leqslant\frac{4}{4+r}+\frac{1}{5}r^2 \tag{5}$$

with equality in each instance if and only if $r=0$. Moreover, the coefficient 1/5 in the second inequality in (5) is the best possible.

1 Preliminaries

In this section, we prove two technical lemmas needed in the proofs of our main results stated in last section. In the sequel, we always let $\mathbf{N}$ be the set of natural numbers, put

$$a_n=\left[\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\right]^2, b_n=\frac{\Gamma(n+1/2)\Gamma(2n+1)}{\Gamma(n+1)\Gamma(2n+1/2)},$$

for $n\in\mathbf{N}\cup\{0\}$, where $\Gamma(x)$ is the classical gamma function[1]255-258, and for $n\in\mathrm{N}$, let

$$\begin{cases}A_1=20a_1-5a_0=0\\ A_2=4(a_0-5a_1+5a_2)=29\pi/16\\ A_{2n+1}=a_{2n-2}+4a_{2n-1}+20a_{2n+1}-5a_n\\ A_{2n+2}=20a_{n+1}-a_{2n-1}-4a_{2n}-20a_{2n+2}\end{cases} \tag{6}$$

Clearly, $a_0=\pi, a_1=\pi/4$ and $a_2=9\pi/64$.

Lemma 1 The sequence $\{b_n\}$ is strictly increasing in $n\in\mathbf{N}$ with $b_1=4/3$ and $\lim_{n\to\infty}b_n=\sqrt{2}$, while the sequence $\{a_n\}$ is strictly decreasing in $n\in\mathbf{N}$ with $\lim_{n\to\infty}a_n=0$. Furthermore, for $n\in\mathbf{N}$,

$$16/9\leqslant a_n/a_{2n}\leqslant 2 \tag{7}$$

Proof. Clearly, $b_1=4/3$. By [1]257, $\lim_{n\to\infty}b_n=\sqrt{2}$. It is easy to verify that

$$\frac{b_{n+1}}{b_n}=\frac{(n+1/2)(2n+2)(2n+1)}{(n+1)(2n+3/2)(2n+1/2)}=1+\frac{1}{16n^2+16n+3}>1,$$

yielding the monotonicity of $\{b_n\}$.

Clearly, $\lim_{n\to\infty}a_n=0$. Since $a_{n+1}/a_n=[(2n+1)/(2n+2)]^2<1$, the result for $\{a_n\}$ follows. It is easy to show that $a_n/a_{2n}=b_n^2$. Hence (7) follows from the result for $\{b_n\}$. □

Lemma 2 For $n\in\mathbf{N}$,

$$0<A_{2n+2}<A_{2n+1} \tag{8}$$

Proof. By (6), $A_{2n+2}>0$ if and only if

$$P_1(n)\equiv\frac{20a_{n+1}}{a_{2n-1}+4a_{2n}+20a_{2n+2}}>1 \tag{9}$$

It is easy to verify that for $n\in\mathbf{N}$,

$$\frac{a_{n+1}}{a_n}=\left[\frac{2n+1}{2(n+1)}\right]^2, \frac{a_{2n-1}}{a_{2n}}=\left(\frac{4n}{4n-1}\right)^2, \frac{a_{2n+2}}{a_{2n}}=\left[\frac{(4n+3)(4n+1)}{8(n+1)(2n+1)}\right]^2 \tag{10}$$

By (7) and (10), we have

$$P_1(n)=\frac{5[(2n+1)/(n+1)]^2}{[4n/(4n-1)]^2+4+5\{(4n+3)(4n+1)/[4(n+1)(2n+1)]\}^2}\cdot\frac{a_n}{a_{2n}}$$
$$\geqslant\frac{16}{9}\cdot\frac{5[(2n+1)/(n+1)]^2}{[4n/(4n-1)]^2+4+5\{(4n+3)(4n+1)/[4(n+1)(2n+1)]\}^2},$$

which is greater than 1 since

$$\frac{16}{9}-\frac{[4n/(4n-1)]^2+4+5\{(4n+3)(4n+1)/[4(n+1)(2n+1)]\}^2}{5[(2n+1)/(n+1)]^2}$$
$$=\frac{97280n^6+95232n^5+6912n^4-22528n^3-9840n^2+72n+299}{720(4n-1)^2(2n+1)^4}$$
$$=\frac{1024n^3(95n^3-22)+48n^2(1984n^3-205)+6912n^4+72n+299}{720(4n-1)^2(2n+1)^4}>0.$$

Hence the first inequality in (8) follows from (9).

Next, by (6), we can easily see that $A_{2n+2}<A_{2n+1}$ if and only if

$$P_2(n)\equiv\frac{5a_n+20a_{n+1}}{a_{2n-2}+5a_{2n-1}+4a_{2n}+20a_{2n+1}+20a_{2n+2}}<1 \tag{11}$$

It is easy to show that

$$\begin{cases}\dfrac{a_{2n-2}}{a_{2n}}=\left[\dfrac{8n(2n-1)}{(4n-3)(4n-1)}\right]^2\\ \dfrac{a_{2n+1}}{a_{2n}}=\left[\dfrac{4n+1}{2(2n+1)}\right]^2\end{cases} \tag{12}$$

For $n\in\mathbf{N}$, let $P_3(n)=5+5[(2n+1)/(n+1)]^2$ and

$$P_4(n)=\left[\frac{8n(2n-1)}{(4n-3)(4n-1)}\right]^2+5\left(\frac{4n}{4n-1}\right)^2+4+5\left(\frac{4n+1}{2n+1}\right)^2+5\left[\frac{(4n+1)(4n+3)}{4(n+1)(2n+1)}\right]^2.$$

Then it follows from (10)-(12) and (7) that

$$P_2(n)=\frac{P_3(n)}{P_4(n)}\cdot\frac{a_n}{a_{2n}}\leqslant 2\,\frac{P_3(n)}{P_4(n)},$$

and hence, $P_2(n)<1$ for all $n\in\mathbf{N}$ if

$$P_5(n)\equiv P_4(n)-2P_3(n)>0, n\in\mathbf{N} \tag{13}$$

By computation, we have

$$16(4n-3)^2(4n-1)^2(2n+1)^2(n+1)^2P_5(n)$$
$$=221184n^7-97280n^6-93184n^5+43648n^4-33792n^3+15536n^2+7392n-1179$$
$$=1024n^5(216n^2-95n-91)+1408n^3(31n-24)+15536n^2+(7392n-1179)>0$$

for $n\in\mathbf{N}$. Hence (13) really holds, so that $A_{2n+2}<A_{2n+1}$ by (11). □

2 Proof of Theorem 1

a) Let $f_1(r)=(4+r)/\sqrt[4]{1+r}$ and $f_2(r)=\sqrt[4]{1+r}\,\mathscr{K}(r)/\mathscr{K}(\sqrt{r})$ for $r\in[0,1)$. Then $f(r)=f_1(r)f_2(r)$, f_2 is strictly increasing from $[0,1)$ onto $[1,\sqrt[4]{2})$ by [6, Theorem 1.1], and it is easy to prove that f_1 is strictly increasing from $[0,1]$ onto $[4,\frac{5}{\sqrt[4]{2}}]$. Hence the result for f follows.

b) Clearly, $g(r)=[f(r)-4]\mathscr{K}(\sqrt{r})$. Hence the result for g follows from part a).

c) The first inequality and its equality case in (5) follows from part a) or part b). (Note that this result has been proved in [7] by using a quite complicated method.)

For $r\in[0,1)$, let $F(a,b;c;r)$ be the Gaussian hypergeometric function[1]556, and $h(r)=10\{[4+r^2(4+r)/5]\mathscr{K}(\sqrt{r})-(4+r)\mathscr{K}(r)\}$. By [1]591,

$$\mathscr{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(1/2,n)}{n!}\right]^2r^{2n}=\frac{1}{2}\sum_{n=0}^{\infty}a_nr^{2n},$$

by which $h(r)$ has the following power series expansion

$$\begin{aligned} h(r) &= (r^3+4r^2+20)\sum_{n=0}^{\infty}a_n r^n - 5(4+r)\sum_{n=0}^{\infty}a_n r^{2n} \\ &= 20\sum_{n=0}^{\infty}a_n r^n + 4\sum_{n=2}^{\infty}a_{n-2}r^n + \sum_{n=3}^{\infty}a_{n-3}r^n - 5\sum_{n=0}^{\infty}a_n r^{2n+1} - 20\sum_{n=0}^{\infty}a_n r^{2n} \\ &= 20\sum_{n=2}^{\infty}a_n r^n + 4\sum_{n=2}^{\infty}a_{n-2}r^n + \sum_{n=3}^{\infty}a_{n-3}r^n - 5\sum_{n=1}^{\infty}a_n r^{2n+1} - 20\sum_{n=1}^{\infty}a_n r^{2n} \\ &= A_2 r^2 + \sum_{n=3}^{\infty}(20a_n+4a_{n-2}+a_{n-3})r^n - 5\sum_{n=1}^{\infty}a_n r^{2n+1} - 20\sum_{n=2}^{\infty}a_n r^{2n} \\ &= A_2 r^2 + \sum_{n=1}^{\infty}(A_{2n+1}-A_{2n+2}r)r^{2n+1} \end{aligned} \tag{14}$$

Since $A_{2n+1}-rA_{2n+2}>A_{2n+1}-A_{2n+2}>0$ for $r\in[0,1)$ and $n\in\mathbf{N}$ by Lemma 2, it follows from (14) and Lemma 2 that

$$h(r)\geqslant A_2 r^2+\sum_{n=1}^{\infty}(A_{2n+1}-A_{2n+2})r^{2n+1}\geqslant 0 \tag{15}$$

with equality in each instance if and only if $r=0$. This yields the second inequality in (5) and its equality case.

It is well known that

$$\mathscr{K}(r)=\log\frac{4}{r'}+O\Big((1-r^2)\log(1-r^2)\Big) \tag{16}$$

as $r\to 1$[1]559. For $r\in[0,1)$, let

$$h_1(r)=r^{-2}\big(\mathscr{K}(r)/\mathscr{K}(\sqrt{r})-4/(4+r)\big).$$

Then $h_1(1^-)=1/5$, since $\lim_{r\to 1}\mathscr{K}(r)/\mathscr{K}(\sqrt{r})=1$ by (16). This shows that the constant 1/5 in (5) is the best possible. □

3 Concluding Remarks

a) The upper bound given in (5) is better than each of known upper bounds of $\mathscr{K}(r)/\mathscr{K}(\sqrt{r})$ given in (3)−(4). In order to show this, for $r\in[0,1)$, we let

$$F_1(r)=\frac{4}{4+r}+\frac{1}{5}r^2,\ F_2(r)=\left(\frac{2}{1+r}\right)^{1/4}\ \text{and}\ F_3(r)=\frac{1}{[(1+r)(1-r^2)]^{1/4}},$$

and give the comparisons between these known upper bounds of $\mathscr{K}(r)/\mathscr{K}(\sqrt{r})$ below.

i) For all $r\in[0,1)$, it is clear that $F_2(r)>1$ and

$$1-F_1(r)=\frac{r(5+r)(1-r)}{5(4+r)}>0,$$

which shows that $F_1(r)<1<F_2(r)$, and hence the upper bound given in (5) is not only better than that given in (4), but also better than the first upper bound given in (3).

ii) One can verify that $F_1(r)<F_3(r)$ for all $r\in[0,1)$. As a matter of fact, we have

$$r^{-2}[625(4+r)^4(1+r)(1-r^2)][F_3(r)^4-F_1(r)^4]=r^{13}+17r^{12}+111r^{11}+415r^{10}+1440r^9+4624r^8+9808r^7+21664r^6+46240r^5+43680r^4+102400r^3+58225r^2+10000r+92000>0.$$

Hence the upper bound given in (5) is better than the second upper bound given in (3).

b) From (15) and Lemma 2, we see that for $r\in[0,1)$,

$$\frac{\mathscr{K}(r)}{\mathscr{K}(\sqrt{r})}\leqslant\frac{4}{4+r}+\frac{1}{5}r^2-\frac{A_2 r^2}{10(4+r)\mathscr{K}(\sqrt{r})} \tag{17}$$

which is better than the second inequality in (5). Unfortunately, the third term of the upper bound in (17) contains $\mathscr{K}(\sqrt{r})$.

c) It can be proved that Theorem 1 can be extended to the generalized elliptic integrals $\mathscr{K}_a(r)\equiv(\pi/2)F(a,1-a;1;r^2)$ for $a\in[0,1)$ and $r\in[0,1)$, although the proof of this extension is more difficult than that of Theorem 1. However, Theorem 1 can not be extended to the zero-balanced hypergeometric functions $F(a,b;a+b;x)$ for some values of $a\in(0,\infty)$ and $b\in(0,\infty)$. These results will be given in a separate paper.

d) Our computation supports the validity of the following conjecture: There exists a unique number $r_0=0.70679\cdots$, such that the function

$$F(r)\equiv r^{-2}\left(\mathscr{K}(r)/\mathscr{K}(\sqrt{r})-4/(4+r)\right)$$

is strictly decreasing on $(0,r_0]$ and increasing on $[r_0,1)$, with $F(0^+)=7/64$, $F(1^-)=1/5$ and $c=F(r_0)=\inf\limits_{0<r<1}F(r)\approx 0.0781$. If this conjecture is true, then

$$\frac{4}{4+r}+cr^2\leqslant\frac{\mathscr{K}(r)}{\mathscr{K}(\sqrt{r})}\leqslant\frac{4}{4+r}+\frac{1}{5}r^2 \tag{18}$$

for $r\in[0,1)$, with equality in each instance if and only if $r=0$, and the coefficients c and $1/5$ in (18) are both the best possible, thus improving the first inequality in (5).

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第一类完全椭圆积分之商的一个双向不等式

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摘要： 建立了第一类完全椭圆积分的商 $\mathcal{K}(r)/\mathcal{K}(\sqrt{r})$ 所满足的一个双向不等式。该不等式给出的上界小于至今已知的所有上界，而下界的证明则简化了最近由 Alzer 和 Richards 给出的证明。

关键词： 完全椭圆积分；单调性；上下界；不等式

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