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Monotonicity and convexity properties of the Gamma and Psi functions

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Abstract: The authors present several monotonicity and log-convexity properties of the gamma function $\Gamma(x)$, and some monotonicity and convexity properties of certain combinations defined in terms of $\Gamma(x)$, the psi function $\psi(x)$, $\psi'(x)$ and $\psi''(x)$, by which several known results are improved.

Key words: gamma function; psi function; monotonicity; convexity; inequality

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0 Introduction

For $n\in\mathbf{N}=\{n|n$ is a positive integer$\}$, let $\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}k^{-1}-\log n\right)=0.57721\cdots$ and $\zeta(s)=\sum_{k=1}^{\infty}k^{-s}$ denote the Euler-Mascheroni constant and the Riemann zeta function, respectively. Throughout this paper, we let $\alpha=75[28\zeta(3)+\pi^3]/64-75=0.77797\cdots$, $\beta=2\zeta(3)/3=0.80137\cdots$ and $\delta=18(3-\gamma-\log\pi-\pi^2/8)=0.79837\cdots$ As usual, for $x,y>0$, the gamma and psi functions are defined as

$$\Gamma(x)=\int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t \text{ and } \psi(x)=\frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)},$$

respectively (cf. [1-4]).

During the past decades, many authors have obtained various properties for the functions $\Gamma(x)$, $\psi(x)$ and its derivatives. (Cf.[3],[5]-[18] and bibliographies there.) For example, in [5, Theorem 1.1 and Lemma 2.1], some inequalities were obtained for the function $f(x)\equiv x\psi(x+1)-\log\Gamma(x+1)$, and it was proved that $g(x)\equiv x^2[\psi'(x+1)+x\psi''(x+1)]$ is strictly increasing from $[0,\infty)$ onto $[0,1/2)$, while the function $x\mapsto g(x)/x$ is not monotone on $(0,\infty)$. In [6-7] and [18], several monotonicity properties and inequalities were obtained for the gamma function $\Gamma(x)$. In [18], it was proved that the function $F(x)\equiv(x+1)^{-1}[\Gamma(x+1)]^{1/x}$ ($G(x)\equiv(x+1)^{-1/2}[\Gamma(x+1)]^{1/x}$) is strictly decreasing (increasing, respectively) on $[1,\infty)$. In [7], it was shown that the function $F(G)$ is strictly decreasing and log-convex(increasing and log-concave, respectively) on $(0,\infty)$ by using complicated methods, and some other properties of $\Gamma(x)$ were derived. Such kind of studies usually rely on analytic properties of $\Gamma(x)$, $\psi(x)$, $\psi^{(n)}(x)$, and those of certain combinations defined in terms of these functions.

The main purpose of this paper is to improve the above-mentioned known conclusions for the functions f, g, F and G, and some other main results proved in [7], by applying recent results for $\Gamma(x)$, $\psi(x)$ and $\psi^{(n)}(x)$.

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1 Preliminaries

In the sequel, we shall frequently apply the following formulas [1, 6.1.40, 6.4.2 & 6.4.11]:

$$\log \Gamma(x) \sim \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} (x \to \infty) \tag{1}$$

$$\psi^{(n)}(1) = (-1)^{n+1} n!\ \zeta(n+1) \tag{2}$$

$$\psi^{(n)}(x) \sim (-1)^{n+1}\left[\frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \frac{(n+1)!}{12x^{n+2}} + \cdots\right](x \to \infty) \tag{3}$$

where B_{2k} for $k \in \mathbf{N}$ are the Bernoulli numbers (see [1, 23.1]).

First, we record the following theorem proved in [19, Theorems 1.1-1.2] and needed in the proofs of our results stated in Section 2.

Theorem A. a) For each $n \in \mathbf{N}$, the function $G_n(x) \equiv (-1)^{n+1}[n\psi^{(n)}(x+1) + x\psi^{(n+1)}(x+1)]$ is completely monotonic on $[0,\infty)$, with $G_n(0) = n!\ n\zeta(n+1)$ and $G_n(\infty) = 0$.

b) Let $f_1(x) = xG_{n+1}(x)/G_n(x)$ for each $n \in \mathbf{N}$ and for $x \in [0,\infty)$. Then for each $n \in \mathbf{N}$ and for all $x \in [0,\infty)$,

$$f_1(0) = 0 = \inf_{0 \leqslant x < \infty} f_1(x) \leqslant f_1(x) < \sup_{0 \leqslant x < \infty} f_1(x) = n+1 = f_1(\infty) \tag{4}$$

c) For $x \in (0,\infty)$, let $f_2(x) \equiv x^{-2}(x+1)f(x)$, where $f(x) = x\psi(x+1) - \log\Gamma(x+1)$. Then f_2 is strictly increasing from $(0,\infty)$ onto $(\pi^2/12, 1)$. In particular,

$$\frac{\pi^2 x^2}{12(x+1)} \leqslant x\psi(x+1) - \log\Gamma(x+1) \leqslant \frac{x^2}{x+1} \tag{5}$$

for all $x \in [0,\infty)$, with equality in each instance if and only if $x = 0$.

d) For $x \in (0,\infty)$, let $f_3(x) = x^{-3}(x+1)^2[2f(x) - x^2\psi'(x+1)]$, $f_4(x) = (x+1)^{-2}f_3(x)$, and put $c_0 = \inf_{x \in (0,\infty)} f_3(x)$. Then $f_3(0^+) = \beta$, $f_3(1/2) = \delta$, f_3 is not monotone on $(0,\infty)$, and f_4 is strictly decreasing and convex from $(0,\infty)$ onto $(0,\beta)$. Furthermore,

$$\alpha \leqslant c_0 < \delta \tag{6}$$

$$c_0 \leqslant f_3(x) < \sup_{x \in (0,\infty)} f_3(x) = f_3(\infty) = 1 \tag{7}$$

and

$$\frac{\alpha x^3}{(x+1)^2} \leqslant \frac{c_0 x^3}{(x+1)^2} \leqslant 2f(x) - x^2\psi'(x+1) \leqslant \frac{x^3}{(x+1)^2}\min\{1, \beta(x+1)^2\} \tag{8}$$

for $x \in (0,\infty)$. Each of the equalities in (8) holds if and only if $x = 0$.

Next, we prove the following theorem, which improves [5, Theorem 1.1 & Lemma 2.1].

Theorem 1. Let G_n and f be as in Theorem A. For real numbers a and b, define the functions $g_{1,a}$ and $g_{2,b}$ on $(0,\infty)$ by

$$g_{1,a}(x) = x^{-a}f(x) \text{ and } g_{2,b}(x) = x^b G_1(x),$$

respectively. Then we have the following conclusions:

a) The function $g_{1,a}$ is strictly increasing (decreasing) on $(0,\infty)$ if and only if $a \leqslant 1$ ($a \geqslant 2$, respectively), with $g_{1,1}((0,\infty)) = (0,1)$ and $g_{1,2}((0,\infty)) = (0, \pi^2/12)$. In particular, for $x \in [0,\infty)$,

$$\frac{\pi^2 x^2}{12(x+1)} \leqslant x\psi(x+1) - \log\Gamma(x+1) \leqslant \min\left\{\frac{\pi^2}{12}x^2, \frac{x^2}{x+1}\right\} \tag{9}$$

with equality in each instance if and only if $x = 0$.

b) The function $g_{2,b}$ is strictly increasing (decreasing) on $(0,\infty)$ if and only if $b \geqslant 2$ ($b \leqslant 0$, respectively), with $g_{2,2}([0,\infty)) = [0, 1/2)$ and $g_{2,0}([0,\infty)) = (0, \pi^2/6]$.

Proof: a) Let $g_1(x) = x^2\psi'(x+1)/f(x)$ for $x \in (0,\infty)$. Since $f'(x) = x\psi'(x+1) > 0$, f is strictly increasing on $(0,\infty)$ and $f(x) > f(0) = 0$ for $x \in (0,\infty)$. By differentiation,

$$x^{a+1}g'_{1,a}(x)/f(x) = g_1(x) - a \tag{10}$$

By [5, Theorem 1.1(4)], we have

$$\inf_{0<x<\infty} g_1(x)=g_1(\infty)=1 \text{ and } \sup_{0<x<\infty} g_1(x)=g_1(0^+)=2 \tag{11}$$

Hence it follows from (10) that

$$g'_{1,a}(x)\leqslant 0 \Leftrightarrow a\geqslant \sup_{0<x<\infty} g_1(x)=2$$

and

$$g'_{1,a}(x)\geqslant 0 \Leftrightarrow a\leqslant \inf_{0<x<\infty} g_1(x)=1.$$

This yields the assertion on the monotonicity of $g_{1,a}$.

By l'Hôpital's rule, (2) and (3), we obtain

$$g_{1,1}(0^+)=\lim_{x\to 0}\frac{f(x)}{x}=0,\ g_{1,1}(\infty)=\lim_{x\to\infty}\frac{f(x)}{x}=1,$$

$$g_{1,2}(0^+)=\lim_{x\to 0}\frac{f(x)}{x^2}=\frac{1}{2}\psi'(1)=\frac{\pi^2}{12},\ g_{1,2}(\infty)=\lim_{x\to\infty}\frac{f(x)}{x}=0.$$

The first inequality and the second upper bound in (9) follow from (5), and the first upper bound in (9) follows from the monotonicity property of $g_{1,2}$. The equality case in (9) is clear.

b) Clearly, $g_{2,2}(0)=0$, $g_{2,0}(0)=\psi'(1)=\zeta(2)=\pi^2/6$ and $g_{2,0}(\infty)=G_1(\infty)=0$. By (3), we obtain the limiting value $g_{2,2}(\infty)=\lim_{x\to\infty}x^2G_1(x)=1/2$.

Let G_1 and f_1, with $n=1$, be as in Theorem A. Then by differentiation,

$$x^{1-b}g'_{2,b}(x)/G_1(x)=b-f_1(x) \tag{12}$$

which yields the assertion on the monotonicity of $g_{2,b}$ by Theorem A(2).

2 Some Properties of the Gamma Function

In [7, Theorem 1] ([7, Theorem 2]), it was proved that the function

$$F(x)\equiv(x+1)^{-1}[\Gamma(x+1)]^{1/x}\,(G(x)\equiv(x+1)^{-1/2}[\Gamma(x+1)]^{1/x})$$

is strictly decreasing and strictly log-convex (increasing and log-concave, respectively) on $(0,\infty)$. Our following theorem improves these known conclusions.

Theorem 2. Let c_0 and α be as in Theorem A, and for each $c\in\mathbf{R}$, define the function F on $(0,\infty)$ by $F(x)=(x+1)^{-c}[\Gamma(x+1)]^{1/x}$. Then we have the following conclusions:

a) F is strictly decreasing on $(0,\infty)$ if and only if $c\geqslant 1$, with $F((0,\infty))=(e^{-1},e^{-\gamma})$ if $c=1$, and $F((0,\infty))=(0,e^{-\gamma})$ if $c>1$. Moreover, F is log-convex on $(0,\infty)$ if and only if $c\geqslant 1$.

b) F is strictly increasing on $(0,\infty)$ if and only if $c\leqslant\pi^2/12$. If $c\leqslant\pi^2/12$, then $F((0,\infty))=(e^{-\gamma},\infty)$.

c) F is log-concave on $(0,\infty)$ if and only if $c\leqslant c_0$. In particular, F is log-concave on $(0,\infty)$ if $c\leqslant\alpha$.

Proof: Let $g_{1,2}$, f_2 and f_3 be as in Theorem 1, Theorem A(c) and Theorem A(d), respectively. Then by differentiation, we obtain

$$\frac{F'(x)}{F(x)}=h_1(x)\equiv g_{1,2}(x)-\frac{c}{x+1} \tag{13}$$

$$\frac{F'(x)}{F(x)}=\frac{f_2(x)-c}{x+1} \tag{14}$$

$$h'_1(x)=\frac{c-f_3(x)}{(x+1)^2} \tag{15}$$

a) By (14) and Theorem A(3), for $x\in(0,\infty)$,

$$F'(x)<0\Leftrightarrow c>f_2(x)\Leftrightarrow c\geqslant\sup_{0<x<\infty}f_2(x)=1,$$

which shows that F is strictly decreasing on $(0,\infty)$ if and only if $c\geqslant 1$.

It follows from (7) and (13) that

$$F \text{ is log-convex on } (0,\infty)\Leftrightarrow h_1 \text{ is strictly increasing on } (0,\infty)\Leftrightarrow c\geqslant\sup_{0<x<\infty}f_3(x)=1.$$

It is well know that $\psi(1)=-\gamma$ (see [1, 6.3.2]). If $c=1$, then by l'Hôpital's rule and (1),

$$F(0^+)=\lim_{x\to 0}[\Gamma(x+1)]^{1/x}=\lim_{x\to 0}\exp\left(\frac{\log\Gamma(x+1)}{x}\right)=\lim_{x\to 0}\exp(\psi(x+1))=e^{\psi(1)}=e^{-\gamma},$$

$$F(\infty)=\lim_{x\to\infty}\exp\left(\frac{\log x+\log\Gamma(x)}{x}-\log(x+1)\right)=\lim_{x\to\infty}\exp\left(\frac{(x+1/2)\log x-x-x\log(x+1)}{x}\right)$$
$$=\lim_{x\to\infty}\exp\left(\frac{\log x}{2x}-1\right)=e^{-1}.$$

Similarly, if $c>1$, then $F(0^+)=e^{-\gamma}$ and

$$F(\infty)=\lim_{x\to\infty}\exp\left(\frac{\log x+\log\Gamma(x)}{x}-c\log(x+1)\right)=\lim_{x\to\infty}\exp\left(\frac{(x+1/2)\log x-x}{x}-c\log(x+1)\right)$$
$$=\lim_{x\to\infty}\exp\left((c-1)\log\frac{1}{x}+\frac{\log x}{2x}-1\right)=0.$$

b) It follows from (14) and Theorem A(3) that for all $x\in(0,\infty)$,

$$F'(x)>0\Leftrightarrow c<f_2(x)\Leftrightarrow c\leqslant f_2(0^+)=\frac{\pi^2}{12},$$

that is, F is strictly increasing on $(0,\infty)$ if and only if $c\leqslant\pi^2/12$.

Clearly, if $c\leqslant\pi^2/12$, then $F(0^+)=e^{-\gamma}$. Since $F(\infty)=e^{-1}$ when $c=1$,

$$F(\infty)=\lim_{x\to\infty}(x+1)^{1-c}\cdot\frac{\Gamma(x+1)^{1/x}}{x+1}=\infty.$$

c) It follows from (13), (15) and Theorem A(d) that on $(0,\infty)$,

$$F \text{ is log-concave}\Leftrightarrow h_1 \text{ is strictly decreasing}\Leftrightarrow c\leqslant f_3(x)\Leftrightarrow c\leqslant\inf_{0<x<\infty}f_3(x)=c_0.$$

The remaining conclusion is clear.

The following corollary improves [7, Corollaries 1-2].

Corollary 3. For $x,y\in(0,\infty)$ with $y\geqslant x$,

$$\frac{x+1}{y+1}\leqslant\frac{\Gamma(x+1)^{1/x}}{\Gamma(y+1)^{1/y}}\leqslant\left(\frac{x+1}{y+1}\right)^{\pi^2/12}\tag{16}$$

with equality in each instance if and only if $y=x$. Moreover, for $x\in(0,\infty)$,

$$h_2(x)^x<\Gamma(x+1)<[e^{-\gamma}(x+1)]^x\tag{17}$$

where $h_2(x)=\max\{e^{-1}(x+1),\ e^{-\gamma}(x+1)^{\pi^2/12}\}$.

Proof: It follows from Theorem 2(a)-(b) that

$$\frac{\Gamma(x+1)^{1/x}}{x+1}\geqslant\frac{\Gamma(y+1)^{1/y}}{y+1}\quad\text{and}\quad\frac{\Gamma(x+1)^{1/x}}{(x+1)^{\pi^2/12}}\leqslant\frac{\Gamma(y+1)^{1/y}}{(y+1)^{\pi^2/12}},$$

with equality in each instance if and only if $y=x$. This yields the double inequality (16) and its equality case.

The double inequality (17) follows from Theorem 2(a)-(b).

Remark. Let h_2 be as in Corollary 3, and $h_3(x)=e^{\gamma-1}(x+1)^{1-\pi^2/12}$ for $x\in(0,\infty)$. Then it is clear that h_3 is strictly increasing from $(0,\infty)$ onto $(e^{\gamma-1},\infty)$, and

$$e^{-1}(x+1)/[e^{-\gamma}(x+1)^{\pi^2/12}]=h_3(x).$$

Hence there exists a unique number $x_1\in(0,\infty)$ such that the function

$$h_2(x)=\begin{cases}e^{-\gamma}(x+1)^{\pi^2/12}, & \text{if } x\in(0,x_1)\\ e^{-1}(x+1), & \text{if } x\in[x_1,\infty)\end{cases}.$$

In [7, Theorems 4-5], it was proved that the function $G(x)\equiv\Gamma(x+1)^{1/x}$ is strictly increasing on $(0,\infty)$, and $H(x)\equiv x^\eta\Gamma(x+1)^{1/x}$ is strictly increasing (decreasing) on $(0,\infty)$ if $\eta\geqslant0$ ($\eta\leqslant-1$, respectively). The following theorem strengthens these results.

Theorem 4. a) The function $G(x)\equiv\Gamma(x+1)^{1/x}$ is strictly increasing and log-concave from $(0,\infty)$ onto $(e^{-\gamma},\infty)$.

b) For each $\eta\in\mathbf{R}$, define the function H on $(0,\infty)$ by $H(x)\equiv x^\eta\Gamma(x+1)^{1/x}$. Then H is strictly increasing(decreasing) on $(0,\infty)$ if and only if $\eta\geqslant0$ ($\eta\leqslant-1$, respectively), with $H((0,\infty))=(e^{-\gamma},\infty)$ if $\eta=0$, and $H((0,\infty))=(1/e,\infty)$ if $\eta=-1$.

Proof: a) By logarithmic differentiation, $G'(x)/G(x)=g_{1,2}(x)$, where $g_{1,2}$ is as in Theorem 1. This yields the monotonicity and log-concavity properties of G by Theorem 1(a).

Clearly, $G(0^+)=e^{-\gamma}$. Applying (1), we can obtain the limiting value $G(\infty)=\infty$.

b) Let $g_{1,1}$ be as in Theorem 1. Then by logarithmic differentiation, $xH'(x)/H(x)=\eta+g_{1,1}(x)$, and hence the assertion on the monotonicity of H follows from Theorem 1(a).

It is clear that $H(x)=G(x)$ if $\eta=0$. Hence $H((0,\infty))=(e^{-\gamma},\infty)$ if $\eta=0$.

If $\eta=-1$, then $H(x)=G(x)/x$, so that $H(0^+)=\infty$. By (1.1), $H(\infty)=e^{-1}$.

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Gamma 函数和 Psi 函数的单调性与凹凸性

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摘要: 给出了 gamma 函数 $\Gamma(x)$的几个单调性和对数-凹凸性,以及由 $\Gamma(x)$、psi 函数 $\psi(x)$及其导数 $\psi'(x)$和 $\psi''(x)$定义的某种组合的单调性与凹凸性,并运用这些结果实质性地改进了关于 $\Gamma(x)$和 $\psi(x)$的几个已知结果。

关键词: gamma 函数;psi 函数;单调性;凹凸性;不等式

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