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Some Properties of Gamma, Beta and Psi Functions

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Abstract: This paper presents monotonicity and convexity properties of some combinations of Gamma function, Beta function and Psi function, and attains asymptotically sharp upper and lower bounds of these important special functions, thus improving and generalizing several known results of these functions.

Key words: Gamma function; Beta function; Psi function; monotonicity; convexity; inequality

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1 Notation and mian results

For real and positive values of x and y, the Gamma, Beta and Psi functions are defined as

$$\Gamma(x)=\int_0^{\infty} t^{x-1}e^{-t}\mathrm{d}x, B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\quad \psi(x)=\frac{\Gamma'(x)}{\Gamma(x)} \tag{1}$$

respectively. For their extensions to complex variables and for their basic properties, the reader is referred to [1-3]. It is well known that these important special functions have many applications to various fields of mathematics and some other disciplines as well as to engineering (cf. [1-11]). Many authors have obtained various properties of these functions (cf. [2-11]).

In this paper, we present some monotonicity and convexity properties of these functions, from which some new asymptotically sharp estimates of them follow. In addition, several known results for these functions are improved and generalized.

In the sequels, we let γ stand for the Euler's constant as usual. Now we state our main results. Our first result generalizes [4, Theorems 1.12(1)-(2)].

Theorem 1.1 (1) For $s>0$, define the function f_1 on $(0,\infty)$ by $f_1(x)=\Gamma(x+s)/[x^s\Gamma(x)]$. Then f_1 is strictly increasing and log-concave (decreasing and log-convex) from $(0,\infty)$ onto $(0,1)$ ($(1,\infty)$) if $0<s<1$ ($s>1$, respectively), and $f_1(x)\equiv 1$ if $s=1$.

(2) The function $f_2(x)\equiv f_1(1/x)$ is log-convex(log-concave) on $(0,\infty)$ if $1/2<s<1$ ($s>1$, respectively), while $f_3(x)\equiv 1/f_1(x)$ is convex on $(0,\infty)$ for each $s\in(0,1)$.

(3) The function $f_4(x)\equiv x[1-f_1(x)^2]$ is strictly increasing from $(0,\infty)$ onto $(0,s(1-s))$ for each $s\in[1/2,1)$. Moreover, for $1/2\leqslant s<1$ and $x\geqslant 1$,

$$\begin{aligned}P(x,s)x^{(1-\gamma)x+s-1}&\leqslant\\ P(x,s)x^s\Gamma(x+s)&\leqslant\Gamma(x+s)\leqslant\\ Q(x,s)x^s\Gamma(x)&\leqslant Q(x,s)x^{x+s-1}\end{aligned} \tag{2}$$

with equality in each instance if and only if $x=1$,

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where $P(x,s)=\max\{\sqrt{1-[s(1-s)/x]},\Gamma(s+1)\}$ and $Q(x,s)=\min\{\sqrt{1-\{[1-\Gamma(s+1)^2]/x\}},\Gamma(s+1)^{1/x}\}$.

Our next result generalizes [4, Theorem 1.16 (1)].

Theorem 1.2 Let $\alpha=e^{-\gamma-\psi(s)}$ and $\beta=e^{[\psi'(1)-\psi'(s)]/2}=e^{[(\pi^2/6)-\psi'(s)]/2}$. Then we have the following conclusions:

(1) The function $f_5(x)\equiv[xB(x,s)]^{1/x}$ is strictly decreasing and convex (increasing and log-concave) from $(0,\infty)$ onto $(1,\alpha)$ ($(\alpha,1)$) if $0<s<1$ ($s>1$, respectively).

(2) The function $f_6(x)\equiv[\log f_5(x)+\psi(s)+\gamma]/x$ is strictly increasing (decreasing) from $(0,\infty)$ onto $(\log\beta,0)$ ($(0,\log\beta)$) if $0<s<1$ ($s>1$, respectively). In particular, for all $x\in(0,\infty)$

$$x\alpha^{-x}\Gamma(x)\Gamma(s)<\Gamma(x+s)<\beta x\alpha^{-x}\Gamma(x)\Gamma(s) \tag{3}$$

if $s\in(0,1)$. If $s\in(1,\infty)$, the two inequalities in (3) are both reversed.

2 Preliminary results

In this section, we establish the following theorem, which generalizes [4, Theorem 2.1 (1)-(3)] and is needed for the proofs of our main results.

Theorem 2.1 (1) For each $s\in(0,\infty)$, define the function F_1 on $(0,\infty)$ by

$$F_1(x)=\psi(x+s)-\psi(x)-(s/x).$$

Then F_1 is strictly decreasing and convex (increasing and concave) from $(0,\infty)$ onto $(0,\infty)$ ($(-\infty,0)$) if $0<s<1$ ($s>1$, respectively), and $F_1(x)\equiv 0$ if $s=1$. Moreover, the function $F_2(x)=F_1(1/x)$ is convex (concave) on $(0,\infty)$ if $0<s<1$ ($s>1$, respectively).

(2) The function $F_3(x)\equiv xF_1(x)$ is strictly decreasing (increasing) from $(0,\infty)$ onto $(0,1-s)$ ($(1-s,0)$) if $0<s<1$ ($s>1$, respectively). In particular, for $x\in(0,\infty)$

$$\frac{s}{x}<\psi(x+s)-\psi(x)<\frac{1}{x} \tag{4}$$

if $0<s<1$. If $s\in(1,\infty)$, the two inequalities in (4) are both reversed.

(3) The function $F_4(x)\equiv x^2F_1(x)$ is strictly increasing (decreasing) from $(0,\infty)$ onto $(0,s(1-s)/2)$ ($(s(1-s)/2,0)$) if $1/2<s<1$ ($s>1$, respectively). In particular, for $x\in(0,\infty)$

$$\frac{s}{x}<\psi(x+s)-\psi(x)<\frac{s}{x}+\frac{s(1-s)}{2x^2} \tag{5}$$

if $1/2<s<1$. If $s\in(1,\infty)$, the two inequalities in (5) are both reversed. F_4 is not always monotone on $(0,\infty)$ at least for some values of $s\in(0,1/2)$.

Proof (1) By [1, 6.4.1],

$$x^2F'_1(x)=x^2[\psi'(x+s)-\psi'(x)]+s=s-\int_0^\infty\frac{x^2te^{-xt}(1-e^{-st})}{1-e^{-t}}\mathrm{d}t \tag{6}$$

Putting $u=xt$, we can rewrite (6) as

$$x^2F_1'(x)=F_5(x)\equiv s-\int_0^\infty ue^{-u}F_6(x,u)\mathrm{d}u \tag{7}$$

where $F_6(x,u)=(1-e^{-su/x})/(1-e^{-u/x})$. Clearly,

$$F_5(0^+)=s-\int_0^\infty ue^{-u}\mathrm{d}u=s-1 \text{ and } F_5(\infty)=s-s\int_0^\infty ue^{-u}\mathrm{d}u=0.$$

It is easy to verify that the function F_6 is strictly decreasing (increasing) in x on $(0,\infty)$ if $0<s<1$ ($s>1$, respectively). Hence F_5 is strictly increasing (decreasing) in x from $(0,\infty)$ onto $(s-1,0)$ $(0,(s-1))$ if $0<s<1$ ($s>1$, respectively), so that the monotonicity of F_1 follows from (7).

Since [1, 6.3.5]

$$\psi(x+1)=\psi(x)+(1/x) \tag{8}$$

$$F_1(x)=\psi(x+s)-\psi(x+1)+[(1-s)/x] \tag{9}$$

so that

$$F_1(0^+)=\lim_{x\to0^+}\{\psi(x+s)-\psi(x+1)+[(1-s)/x]\}=\infty(-\infty),$$

if $0<s<1$ ($s>1$, respectively). On the other hand, by [1, 6.3.18],

$$\psi(x)\sim\log x-\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\frac{1}{252x^6}+\cdots \tag{10}$$

as $x\to\infty$, and hence $F_1(\infty)=\lim_{x\to\infty}\log(1+s/x)=0$.

If $0<s<1$, then $-F'_1(x)=(1/x^2)\cdot[-F_5(x)]$, which is a product of two positive and decreasing functions, and hence F_1 is convex on $(0,\infty)$. If $s>1$, then $F'_1(x)=(1/x^2)\cdot F_5(x)$, which is a product of two positive and decreasing functions, and hence the concavity of F_1 on $(0,\infty)$ follows.

Next, since $F'_2(x) = -F_5(1/x)$, the conclusion for F_2 follows from the monotonicity of F_5.

(2) Set $y=1/x$. Then

$$F_3(x)=\frac{F_2(y)}{y},\quad \frac{F'_2(y)}{\frac{\mathrm{d}}{\mathrm{d}y}(y)}=F'_2(y),$$

and hence the monotonicity of F_3 follows from the convexity and concavity of F_2 and the Monotone l'Hôpital's Rule (cf. [3, Theorem 1.25]).

By (8), we can rewrite $F_3(x)$ as

$$F_3(x)=x[\psi(x+s)-\psi(x+1)]+(1-s),$$

so that $F_3(0^+)=1-s$. Since

$$\lim_{x\to\infty} x\log\frac{x+s}{x+1}=\lim_{x\to 0}\frac{\log[(1+sx)/(1+x)]}{x}=s-1,$$

by l'Hôpital's Rule, it follows from (10) that

$$F_3(\infty)=\lim_{x\to\infty}x\left[\log\frac{x+s}{x+1}-\frac{1}{2(x+s)}+\frac{1}{2(x+1)}+\mathrm{O}\left(\frac{1}{x^2}\right)\right]+1-s=0.$$

(3) Let $y=1/x$, $F_7(y)=y^2$. Then $F_4(x)=F_2(y)/F_7(y)$, $F_2(0^+)=F_7(0)=0$, $F'_2(0^+)=F'_7(0)=0$,

$$\frac{F'_2(y)}{F'_7(y)}=-\frac{F_5(1/y)}{2y}\tag{11}$$

and by (7),

$$\frac{F''_2(y)}{F''_7(y)}=\frac{F'_5(1/y)}{2y^2}=\frac{1}{2}x^2F'_5(x)=-\frac{1}{2}\int_0^\infty ue^{-u}F_8(x,u)\,\mathrm{d}u\tag{12}$$

where

$$F_8(x,u)=x^2\frac{\partial F_6}{\partial x}=u\frac{e^{-u/x}(1-e^{-su/x})-se^{-su/x}(1-e^{-u/x})}{(1-e^{-u/x})^2}\tag{13}$$

Set $t=u/x$. Then (13) can be rewritten as

$$\frac{1}{u}F_8(x,u)=F_9(t)\equiv\frac{e^{-t}(1-e^{-st})-se^{-st}(1-e^{-t})}{(1-e^{-t})^2}=\frac{e^t-se^{(2-s)t}+(s-1)e^{(1-s)t}}{(e^t-1)^2}\tag{14}$$

Differentiation gives

$$(e^t-1)^3e^{-t}F'_9(t)=h_1(t)\equiv s^2e^{(2-s)t}-e^t+(1+2s-2s^2)e^{(1-s)t}+(1-s)^2e^{-st}-1\tag{15}$$

$$e^{(s-1)t}h'_1(t)=h_2(t)\equiv s^2(2-s)e^t-e^{st}-s(1-s)^2e^{-t}+(1-s)(1+2s-2s^2)\tag{16}$$

$$s^{-1}e^{-t}h'_2(t)=h_3(t)\equiv(1-s)^2e^{-2t}-e^{(s-1)t}+s(2-s)\tag{17}$$

and

$$e^{2t}h'_3(t)=h_4(t)\equiv(1-s)h_5(t)\tag{18}$$

where $h_5(t)=e^{(1+s)t}-2(1-s)$. Clearly, $h_1(0)=h_2(0)=h_3(0)=0$ for all $s\in(0,\infty)$, $h_3(\infty)=s(2-s)>0$ if $0<s<1$, $h_3(\infty)=-\infty$ if $s>1$, and h_5 is strictly increasing from $(0,\infty)$ onto $(2s-1,\infty)$.

Now we study the monotonicity of F_4 by discussing three cases.

Case(i) $s>1$.

In this case, it follows from (18) that h_3 is strictly decreasing from $(0,\infty)$ onto $(-\infty,0)$, so that by (17), h_2 is strictly decreasing on $(0,\infty)$. Therefore by (16), h_1 is strictly decreasing on $(0,\infty)$. This, together with (13)−(15), implies that F_8 is strictly increasing in x on $(0,\infty)$. Hence by [3, Theorem 1.25], the monotonicity of F_4 follows from (11)−(12).

Case(ii) $1/2<s<1$.

In this case, $h_4(t)>0$ by (18). Hence h_3 is strictly increasing from $(0,\infty)$ onto $(0,s(2-s))$. This shows that h_2 is positive and strictly increasing on $(0,\infty)$, and so is h_1. Thus by (14)−(15), F_8 is strictly decreasing in x on $(0,\infty)$, so that F_4 is strictly increasing on $(0,\infty)$ by (11)−(12) and [3, Theorem 1.25].

Case(iii) $0<s<1/2$.

By (8), $F_4(x)$ can be rewritten as

$$F_4(x)=x^2[\psi(x+s)-\psi(x+1)]+(1-s)x,$$

and hence

$$F'_4(x)=2x[\psi(x+s)-\psi(x+1)]+x^2[\psi'(x+s)-\psi'(x+1)]+(1-s).$$

It is easy to obtain the limiting values

$$F'_4(0^+)=(1-s)>0 \text{ and } F'_4(\infty)=0\tag{19}$$

By (8) and [1, 6.3.2, 6.4.2 & 23.2.24],

$$\begin{aligned}F'_4(1)=h_6(s)&\equiv2[\psi(1+s)-\psi(2)]+[\psi'(1+s)-\psi'(2)]+(1-s)=\\&2\psi(1+s)+\psi'(1+s)-2(1-\gamma)-[\psi'(1)-1]+1-s=\\&2\psi(1+s)+\psi'(1+s)-s+2\gamma-(\pi^2/6),\end{aligned}$$

with $h_6(0)=0$ and

$$h'_6(s)=2\psi'(1+s)+\psi''(1+s)-1.$$

By [1, 6.4.2, 23.2.24 & Table 23.3], we have

$$h'_6(0)=2\psi'(1)+\psi''(1)-1=(\pi^2/3)-2\zeta(3)-1=-0.114\,24\ldots,$$

and hence there exists an $s_0\in(0,1/2)$ such that $F'_4(1)=h_6(s)<h_6(0)<0$ for $s\in(0,s_0)$. Hence F_4' changes its sign on $(0,\infty)$. This yields the assertion for F_4 in the case $0<s<1/2$.

3 Proof of the main results

In this section, we prove the main theorems stated in Section 1.

3.1 Proof of Theorem 1.1 (1) Clearly, $f_1(x)\equiv 1$ if $s=1$. Since

$$f_1(0^+)=\lim_{x\to 0}x^{1-s}\cdot\frac{\Gamma(x+s)}{\Gamma(x+1)}=\Gamma(s)\lim_{x\to 0}x^{1-s},$$

$f_1(0^+)=0$ if $0<s<1$, and $f_1(0^+)=\infty$ if $s>1$. Since [1, 6.1.37]

$$\Gamma(x)\sim\sqrt{2\pi}e^{-x}x^{x-(1/2)}\quad\text{as } x\to\infty \tag{20}$$

we obtain the limiting value $f_1(\infty)=1$.

Logarithmic differentiation gives

$$f'_1(x)/f_1(x)=\psi(x+s)-\psi(x)-(s/x),$$

and hence the conclusions for f_1 follow from Theorem 2.1(1).

(2) Let F_4 be as in Theorem 2.1(3). Then $f'_2(x)/f_2(x)=-F_4(1/x)$, and the conclusion for f_2 follows from Theorem 2.1(3).

Since $f'_3(x)=-f'_1(x)/[f_1(x)]^2=-F_1(x)/f_1(x)$, we obtain the result for f_3 from part (1) and Theorem 2.1(1).

(3) Let $y=1/x$, $G_1(y)=1-[f_1(1/y)]^2$ and $G_2(y)=y$ for each $s\in[1/2,1)$, and let F_4 be as in Theorem 2.1(3). Then $G_1(0^+)=G_2(0)=0$ by part (1), $f_4(x)=G_1(y)/G_2(y)$ and

$$G'_1(y)/G'_2(y)=2y^{-2}f_1(1/y)f'_1(1/y)=2[f_1(1/y)]^2F_4(1/y).$$

Hence the monotonicity of f_4 follows from part (1), Theorem 2.1(3) and [3, Theorem 1.25].

By part (1), $f_4(0^+)=0$. Applying l'Hôpital's Rule, we obtain

$$f_4(\infty)=\lim_{y\to 0}\frac{G'_1(y)}{G'_2(y)}=2\lim_{y\to 0}[f_1(1/y)]^2F_4(1/y)=2\lim_{x\to\infty}[f_1(x)]^2F_4(x)=s(1-s).$$

Next, it follows from the log-convexity of f_2 on $(0,1)$ that

$$\log\Gamma(s+1)\leqslant\log f_2(x)\leqslant x\log\Gamma(s+1),$$

for $x\in(0,1]$, or equivalently,

$$\Gamma(s+1)\leqslant f_1(x)\leqslant\Gamma(s+1)^{1/x},$$

for $x\in[1,\infty)$, with equality in each instance if and only if $x=1$. This double inequality yields

$$x^s\Gamma(s+1)\Gamma(x)\leqslant\Gamma(x+s)\leqslant x^s\Gamma(s+1)^{1/x}\Gamma(x) \tag{21}$$

for $x\in[1,\infty)$, with equality in each instance if and only if $x=1$. On the other hand, it follows from the monotonicity of f_4 on $[1,\infty)$ that

$$1-f_1(1)^2\leqslant x[1-f_1(x)^2]<s(1-s),$$

for $x\in[1,\infty)$, and hence

$$x^s\Gamma(x)\sqrt{1-[s(1-s)/x]}\leqslant\Gamma(x+s)\leqslant x^s\Gamma(x)\sqrt{1-\{[1-\Gamma(s+1)^2]/x\}} \tag{22}$$

for $x\in[1,\infty)$, with equality in each instance if and only if $x=1$. Combining (21) and (22), we obtain the second and third inequalities in (2). The first and fourth inequalities in (2) hold by [5, Theorem 1.5]. The equality case is clear.

3.2 Proof of Theorem 1.2 (1) $f_5(x)$ can be rewritten as

$$f_5(x)=[\Gamma(x+1)\Gamma(s)/\Gamma(x+s)]^{1/x}.$$

Logarithmic differentiation gives

$$f'_5(x)/f_5(x)=g(x)\equiv g_1(x)/g_2(x) \tag{23}$$

where $g_1(x)=x[\psi(x+1)-\psi(x+s)]+\log\Gamma(x+s)-\log\Gamma(s)-\log\Gamma(x+1)$ and $g_2(x)=x^2$ with $g_1(0)=g_2(0)=0$, and

$$\frac{g_1'(x)}{g_2'(x)}=g_3(x)\equiv\frac{1}{2}[\psi'(x+1)-\psi'(x+s)] \tag{24}$$

with $g_3(0)=[\psi'(1)-\psi'(s)]/2=[(\pi^2/6)-\psi'(s)]/2$ by [1, 6.4.2 & 23.2.24]. Since ψ' is strictly decreasing on $(0,\infty)$, $g_3(x)<0$ if $0<s<1$ and $g_3(x)>0$ if $s>1$. Since ψ'' is strictly increasing on $(0,\infty)$, $g_3'(x)>0$ if $0<s<1$ and $g_3'(x)<0$ if $s>1$, so that by [3, Theorem 1.25], g is negative and increasing (positive and decreasing) on $(0,\infty)$ if $0<s<1$ ($s>1$, respectively). Therefore for $0<s<1$, $f_5'(x)<0$ and $-f_5'(x)=f_5(x)[-g(x)]$ is a product of two positive and decreasing functions, so that f_5 is convex on $(0,\infty)$. If $s>1$, then $f_5'(x)=f_5(x)\,g(x)>0$ and $f_5'(x)/f_5(x)=g(x)$ is strictly decreasing for $x\in(0,\infty)$, and hence the monotonicity and log-concavity of f_5 follow. By l'Hôpital's Rule,

$$\lim_{x\to 0}\log f_5(x)=\lim_{x\to 0}\frac{1}{x}[\log\Gamma(x+1)-\log\Gamma(x+s)+\log\Gamma(s)]=\lim_{x\to 0}[\psi(x+1)-\psi(x+s)]=-\gamma-\psi(s),$$

and hence $f_5(0^+)=e^{-\gamma-\psi(s)}=\alpha$. By [1, 6.1.40], we have

$$\lim_{x\to\infty}\log f_5(x)=\lim_{x\to\infty}\frac{1}{x}[\log\Gamma(x+1)-\log\Gamma(x+s)+\log\Gamma(s)]=\lim_{x\to\infty}\frac{1}{x}\left[\left(x+\frac{1}{2}\right)\log(x+1)-\left(x+s-\frac{1}{2}\right)\log(x+s)-1+s\right]=\lim_{x\to\infty}\left[\left(1+\frac{1}{2x}\right)\log(x+1)-\left(1+\frac{2s-1}{2x}\right)\log(x+s)\right]=\lim_{x\to\infty}\log\frac{x+1}{x+s}=0,$$

so that $f_5(\infty)=1$.

(2) Let $g_4(x)=\log f_5(x)+\psi(x)+\gamma$ and $g_5(x)=x$. Then $g_4(0^+)=g_5(0)=0$, $f_6(x)=g_4(x)/g_5(x)$ and $g_4'(x)/g_5'(x)=g(x)$, where g is as in (23). Hence the monotonicity of f_6 follows from the property of g above-mentioned.

Clearly, $f_6(\infty)=0$. By l'Hôpital's Rule and (24), $f_6(0^+)=g(0^+)=g_3(0^+)=\log\beta$. The remaining conclusions are clear.

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Gamma、Beta 与 Psi 函数的几个性质

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摘　要：研究揭示了 Gamma、Beta 与 Psi 函数的一些组合的单调性和凹凸性等性质，并据此获得了这些重要特殊函数的渐进精确的上下界，从而改进和推广了关于这些函数的几个已知结果。

关键词：Gamma 函数；Beta 函数；Psi 函数；单调性；凹凸性；不等式

（责任编辑：康　锋）