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On the Best Case of a Kind of Upper Bounds for φ_K-Distortion Function

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Abstract: This paper studies the optimal index α of of $M(r)=\frac{2}{\pi}(r')^2\kappa(r)\kappa'(r)+\log r$ in upper bound estimation in famous Hübner inequation in quasiconformal theory, gains estimated values of upper and lower bounds when max $\{c$: inequality $M(r)<(r')^c \log 4$ is established for all $r\in(0,1)\}$, and proves min $\{c$: $M(r)>(1-r)^c \log 4$ is established for all $r\in(0,1)\}=1$. Thus, very important upper bound of Hersch-Pfluger deviation function $\varphi_K(r)$ in quasiconformal theory and corresponding explicit quasiconformal Schwarz lemma are improved.

Key words: upper bound estimation; Hübner's inequality; Hersch-Pfluger Distortion function; quasiconformal Schwarz lemma

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0 Main Results

The complete elliptic integrals of the first and second kinds are defined by[1-3],

$$\begin{cases}\kappa=\kappa(r)=\int_0^{\pi/2}(1-r^2\sin^2 x)^{-1/2}\mathrm{d}x,\\ \kappa'=\kappa'(r)=\kappa(r'),\\ \kappa(0)=\frac{\pi}{2},\kappa(1)=\infty\end{cases}$$

and

$$\begin{cases}\varepsilon=\varepsilon(r)=\int_0^{\pi/2}(1-r^2\sin^2 x)^{1/2}\mathrm{d}x,\\ \varepsilon'=\varepsilon'(r)=\varepsilon(r'),\\ \varepsilon(0)=\frac{\pi}{2},\varepsilon(1)=1.\end{cases}$$

for $r\in[0,1]$ and $r'=\sqrt{1-r^2}$, respectively. It is well known that the complete elliptic integrals have many important applications in physics, engineering, geometric function theory, quasiconformal analysis, theory of mean values, number theory and many other fields. For these, and for the properties of $\kappa(r)$ and $\varepsilon(r)$, see[1-3, 5-6] and bibliographies there.

Throughout this paper, we let $r'=\sqrt{1-r^2}$ for $r\in[0,1]$, arth be the inverse of the hyperbolic tangent function, and let

$$m(r)=\frac{2}{\pi}(r')^2\kappa\kappa' \text{ and } M(r)=m(r)+\log r \tag{1}$$

for $r\in[0,1]$ with $m(0)=\infty, M(0)=\lim_{r\to 0}=\log 4$ and $M(1)=0$. The functions $m(r)$ and $M(r)$ play a very important role in the study of the distortion functions in quasiconformal theory. (See[3, 5-6, 8-13], for example.)

Let $\mu(r)$ be the modulus of the Grötzsch ring $B^2\backslash[0,r]$ for $0<r<1$, which has the well-known expression[2-4]

$$\mu(r)=\frac{\pi}{2}\frac{\kappa'(r)}{\kappa(r)}.$$

For $r\in[0,1]$ and $K\in(0,\infty)$, the Hersch-Pfluger distortion function $\varphi_K(r)$ is defined by[3-4]

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$$\varphi_K(r)=\mu^{-1}(\mu(r)/K),\ \varphi_K(0)=\varphi_K(1)-1=0 \tag{2}$$

The functions $\mu(r)$ and $\varphi_K(r)$ are two of the most important quasiconformal special functions, which are indispensable to the quasiconformal theory.

In 1952, Hersch and Pfluger extended the classical Schwarz lemma for analytic functions to the class $QC_K(B^2)=\{f \mid f$ is a K-quasiconformal mapping of the unit disk B^2 into itself with $f(0)=0\}$, and established the following well-known quasiconformal Schwarz lemma[7]

$$|f(z)|\leqslant\varphi_K(|z|),\ \text{for } f\in QC_K(B^2) \text{ and } z\in B^2 \tag{3}$$

In addition, other important distortion functions in quasiconformal theory are all defined in terms of $\varphi_K(r)$, and many important distortion properties of quasiconformal mappings are expressed in terms of $\varphi_K(r)$. (cf. [2-4]) It also has applications in some other fields of mathematics (cf. [3, 6, 11]). For example, $\varphi_{1/K}(r)$ is the solution to the classical Ramanujan modular equations[3, 6, 11].

Many properties have been obtained for $\varphi_K(r)$ by applying Hübner's sharp inequality[8]

$$\varphi_K(r)\leqslant r^{1/K}\exp\{(1-1/K)M(r)\},$$

for $K\geqslant 1$ and $0<r<1$. (cf. [3-6, 8-13]) In [9, Corollary 1], it was proved that for all $r\in(0,1)$, $K\in(1,\infty)$ and for real function $g(r)$ on $(0,1)$, the inequality

$$\varphi_K(r)<r^{1/K}\exp\{(1-1/\mathrm{K})g(r)\}$$
$$(\varphi_{1/K}(r)<r^K\exp\{(1-K)g(r)\})$$
$$\text{holds iff } M(r)\leqslant g(r)$$
$$(M(r)\geqslant g(r),\text{respectively}) \tag{4}$$

Therefore, in order to obtain the sharp upper bounds for $\varphi_K(r)$ and for $\varphi_{1/K}(r)$ in terms of elementary functions, one needs only to the sharp lower and upper bounds for the function $M(r)$. Several such kinds of inequalities have been obtained. For example, [9, Theorem 1] presented the following result

$$(1-r)\log 4<\frac{(1-r)\operatorname{arth}\sqrt{r}}{\sqrt{r}}\log 4<M(r)<$$
$$(r')^2\,\frac{\operatorname{arth} r}{r}\log 4<(r')^{4/3}\log 4 \tag{5}$$

for all $r\in(0,1)$, which yields the upper bounds for $\varphi_K(r)$ and for $\varphi_{1/K}(r)$:

$$\varphi_K(r)<4^{(r')^{4/3}(1-1/K)}r^{1/K} \text{ and } \varphi_{1/K}<4^{(1-r)(1-K)}r^K \tag{6}$$

for $r\in(0,1)$ and $K\in(1,\infty)$, while [13, Theorem 1.2] gives the following bound for $\varphi_K(r)$

$$\varphi_K(r)<4^{(r')^{3/2}(1-1/K)}r^{1/K} \tag{7}$$

for $r\in(0,1)$ and $K\in(1,\infty)$ by establishing the inequality $M(r)<(r')^{3/2}\log 4$. These results encourage people to obtain the following type inequalities

$$\varphi_K(r)<4^{(r')^c(1-1/K)}r^{1/K} \text{ and } \varphi_{1/K}(r)<$$
$$4^{(1-r)^c(1-K)}r^K,\ \text{for } r\in(0,1) \text{ and } K\in(1,\infty) \tag{8}$$

with constant $c>0$. By these researches on such kind of inequalities as (8), the following question is natural and significant.

Question 1.1 What is the value of

$\alpha\equiv\max\{$c: The first inequality in (8) holds for all $r\in[0,1]$ and $K\in(1,\infty)\}$?

By [9, Theorem 6 & Corollary 1], Question 1.1 is equivalent to the following

Question 1.2 What is the value of

$$\alpha\equiv\max\{c: M(r)\leqslant (r')^c\log 4 \text{ for all } r\in[0,1]\}? \tag{9}$$

Similarly, another natural question is the following.

Question 1.3 What is the value of

$$\beta\equiv\min\{c: M(r)\geqslant (1-r)^c\log 4 \text{ for all } r\in[0,1]\}? \tag{10}$$

The main purpose of this paper is to study Questions 1.2 and 1.3. We shall obtain the lower and upper bounds for the constant α given in (9), and prove that $\beta=1$ in (10). We now state our main result.

Theorem 1.4 (1) Let α and β be as in (9) and (10), respectively. Then

$$1.6\leqslant\alpha<1.625 \text{ and } \beta=1. \tag{11}$$

In particular,

$$(1-r)^\beta\log 4<M(r)<(r')^{8/5}\log 4 \tag{12}$$

for all $r\in(0,1)$, with the most possible constant $\beta=1$. However for $\alpha\geqslant 1.625$ ($0<\beta<1$), the second (first, respectively) in (12) can be reversed for some values of $r\in(0,1)$.

(2) For all $r\in(0,1)$ and $K\in(1,\infty)$,

$$\varphi_K(r)<4^{(r')^{8/5}(1-1/K)}r^{1/K} \quad (13)$$

and the exponent 1 of $(1-r)$ in second inequality in (6) is the most possible.

(3) For each $f\in QC_K(B^2)$ and $z\in B^2$,

$$|f(z)|\leqslant 4^{(1-|z|^2)^{4/5}(1-1/K)}|z|^{1/K} \quad (14)$$

1 Preliminary Results

In this section, we establish a technical lemma, which is needed in the proof of our main result stated in Section 0. First, we record the following well-known derivative formulas:

$$\frac{d\kappa}{dr}=\frac{\varepsilon-(r')^2\kappa}{r(r')^2},\frac{d\varepsilon}{dr}=\frac{\varepsilon-\kappa}{r} \quad (15)$$

$$\frac{d}{dr}(\kappa-\varepsilon)=\frac{r\varepsilon}{(r')^2},\frac{d}{dr}[\varepsilon-(r')^2\kappa]=r\kappa \quad (16)$$

$$\frac{d}{dr}M(r)=\frac{4}{\pi r}\kappa'(\varepsilon-\kappa)=\frac{4}{\pi r}\left(\frac{\pi}{2}-\kappa\varepsilon'\right) \quad (17)$$

for $0<r<1$. (cf. [2] or [6, Theoren 4. 1]) . Next, we prove the following lemma.

Lemma 2. 1 (1) There exists a unique number $r_1\in(0.723, 0.724)$ such that the function $f_1(r)\equiv\kappa/\sqrt{r}$ is strictly decreasing on $(0,r_1]$, and increasing on $[r_1,1)$.

(2) The function $f_2(r)\equiv(r')^{2/5}\kappa\kappa'$ is strictly decreasing on $(0,1/\sqrt{2}]$.

(3) The function $f_3(r)\equiv(r')^{0.419}(\kappa-\varepsilon)/r^2$ is strictly decreasing on $[0.987,1)$.

(4) The function $f_4(r)\equiv\kappa'/(r')^{\delta}$ is strictly increasing on $[0.987,1)$, where $\delta=0.019$.

Proof (1) Differentiation gives

$$2r^{3/2}(r')^2f'_1(r)=f_5(r)\equiv 2[\varepsilon-(r')^2\kappa]-(r')^2\kappa,$$

which is strictly increasing from $(0,1)$ onto $(-\pi/2,2)$ by [6, Lemmas 5. 2(1) & 5. 4(1)]. Since $f_5(0.723)=-0.004\,74\ldots<0$ and $f_5(0.724)=0.000\,06\ldots>0$, f_5 has a unique zero $r_1\in(0.723, 0.724)$ such that $f_5(r)<0$ for $r\in(0,r_1)$ and $f_5(r)>0$ for $r\in(r_1,1)$. Hence the piecewise monotonicity of f_1 follows.

(2) By differentiation, we have

$$5r(r')^{8/5}f'_2(r)=5[f_6(r)-f_6(r')]-2r^2\kappa\kappa'< 5[f_6(r)-f_6(r')]\leqslant 0$$

for $r\in(0,1/\sqrt{2}]$, since the function $f_6(r)\equiv r^2\kappa'\cdot[\varepsilon-(r')^2\kappa]/r^2$ is strictly increasing from $(0,1)$ onto $(0,\pi/2)$ by [6, Lemmas 5. 2(1) & 5. 4(1)]. This yields the monotonicity of f_2 on $(0,1/\sqrt{2}]$.

(3) Let $c=1.581$ and

$$f_7(r)=\frac{(2-r^2)\varepsilon-2(r')^2\kappa}{r^2(\kappa-\varepsilon)}\ for\ r\in(0,1).$$

Then

$$\frac{r(r')^c}{\kappa-\varepsilon}f'_3(r)=f_7(r)-0.419 \quad (18)$$

by [14, (2. 53)]. Since f_7 is strictly decreasing from $(0,1)$ onto $(0,3/4)$ by [14, Theorem 14], and since $f_7(0.987)=0.4198\ldots$, the monotonicity of f_3 on $[0.987,1)$ follows from (18).

(4) Differentiation gives

$$r(r')^{\delta+2}f'_4(r)=f_8(r)\equiv(\delta+1)r^2\kappa'-\varepsilon' \quad (19)$$

By [6, Lemmas 5. 2(1) & 5. 4(1)], $f_8(r)=\delta r^2\kappa'-(\varepsilon'-r^2\kappa')$ is strictly increasing on $(0,1)$. Since $f_8(0.987)=0.008\,91\ldots>0$, the monotonicity of f_4 on $[0.987, 1)$ follows from (19).

Lemma 2. 2 Let $g_1(r)=(r')^{2/5}\kappa'(\kappa-\varepsilon)/r^2$ for $r\in(0,1)$. Then g_1 is strictly decreasing on $[0.99, 1)$, and

$$g_1(r)<(2\pi\log 4)/5 \text{ for } r\in[0.987, 1) \quad (20)$$

Proof Differentiation gives

$$\frac{5r(r')^{8/5}}{\kappa\kappa'}g'_1(r)=g_2(r)\equiv 5-\left(10-8r^2+5\frac{\varepsilon'}{\kappa'}\right)\frac{\kappa-\varepsilon}{r^2\kappa} \quad (21)$$

By [6, Lemma 5. 2(3)], it holds that

$$g_2(r)<G_1(a,b)\equiv 5-\left[10-8b^2+5\frac{\varepsilon'(a)}{\kappa'(a)}\right]\frac{\kappa(a)-\varepsilon(a)}{a^2\kappa(a)} \quad (22)$$

for each $[a, b]\subset(0, 1)$. By computation, we have:

$G_1(0.99, 0.991)=-0.001\,4\ldots$, $G_1(0.991, 0.993)=-0.029\,1\ldots$, $G_1(0.993, 1)=-0.020\,9\ldots$. Hence it follows from (21) and (22) that g_1 is strictly decreasing on $[0.99, 1)$, so that

$$g_1(r)\leqslant g_1(0.991)=1.712\,1\ldots< \frac{2\pi}{5}\log 4=1.742\,06\ldots, \text{ for } r\in[0.991, 1) \quad (23)$$

Clearly, $g_3(r)\equiv\frac{2\pi}{5}\log 4-g_1(r)=\frac{2\pi}{5}\log 4-f_3(r)f_4(r)$, where f_3 and f_4 are as in Lemma 2. 1. Hence, by Lemma 2. 1,

$$g_3(r)>G_2(a,b)\equiv\frac{2\pi}{5}\log 4-f_3(a)f_4(b) \quad (24)$$

for each $[a, b] \subset (0,1)$. Since $G_2(0.987, 0.991) = 0.0260\ldots$,

$$g_3(r) > 0, \text{ for } r \in [0.987, 0.991]. \quad (25)$$

The inequality (20) now follows from (23) and (25).

2 Proof of the main results

In this section, we prove our main result, which is Theorem 1.4 stated in Section 0.

(1) First, we prove the following inequality

$$M(r) < (r')^{1.6} \log 4 \text{ for } r \in (0,1) \quad (26)$$

which yields the lower bound of α in (11). For this, we let g_1 be as in Lemma 2.2, and let

$$h(r) = M(r) - (r')^{8/5} \log 4$$

for $r \in (0,1)$. Then

$$\frac{\pi}{4r}(r')^{2/5} h'(r) = h_1(r) \equiv \frac{2\pi}{5}\log 4 - g_1(r) \quad (27)$$

by (17). Now we divide the proof of (26) into three steps.

Step 1 We prove that h is strictly decreasing on $(0, 0.53]$.

For this purpose, we let f_2 be as in Lemma 2.1(2) and $h_2(r) = (\kappa - \varepsilon)/(r^2\kappa)$. Then h_2 is strictly increasing from $(0,1)$ onto $(1/2,1)$ by [6, Lemma 5.2(3)]. Hence it follows from (27) and Lemma 2.1(2) that

$$h_1(r) < H_1(a,b) \equiv \frac{2\pi}{5}\log 4 - h_2(a) f_2(b) \quad (28)$$

for each $[a,b] \subset (0,1/\sqrt{2}]$. Computation gives:

$H_1(0,0.48) = -0.0012\ldots$, $H_1(0.48,0.52) = -0.0029\ldots$, $H_1(0.52,0.53) = -0.0014\ldots$.

Hence by (27) and (28), h is strictly decreasing on $[0, 0.53)$ as desired.

Step 2 We prove that h is strictly increasing on $[0.54, 1)$.

It follows from (27) that

$$\frac{\pi}{4r}(r')^{3/4} h'(r) = h_3(r) \equiv \frac{2\pi\log 4}{5}(r')^{7/20} - h_4(r) \quad (29)$$

where $h_4(r) = (r')^{3/4}\kappa'(\kappa - \varepsilon)/r^2$. By [14, Theorem 15], h_4 is strictly decreasing from $(0,1)$ onto $(0, \infty)$. Therefore, it follows from (29) that

$$h_3(r) > H_2(a,b) \equiv \frac{2\pi\log 4}{5}(b')^{7/20} - h_4(a) \quad (30)$$

for each $[a, b] \subset (0,1)$. By computation, we obtain:

$H_2(0.54, 0.545) = 0.0026\ldots$, $H_2(0.545, 0.552) = 0.0060\ldots$, $H_2(0.552, 0.57) = 0.0068\ldots$,

$H_2(0.57, 0.62) = 0.0037\ldots$, $H_2(0.62, 0.705) = 0.0086\ldots$, $H_2(0.705, 0.81) = 0.0034\ldots$,

$H_2(0.81, 0.882) = 0.0089\ldots$, $H_2(0.882, 0.92) = 0.0131\ldots$, $H_2(0.92, 0.94) = 0.0153\ldots$ and $H_2(0.94, 0.955) = 0.0018\ldots$. Hence by (29) and (30), we have:

$$h'(r) > 0 \text{ for } r \in [0.54, 0.955] \quad (31)$$

By [14, (2.53)], one can rewrite (21) as

$$\frac{(r')^2 g'_1(r)}{r g_1(r)} = h_5(r) \equiv h_6(r) - h_7(r) + \frac{3}{5} \quad (32)$$

where

$$h_6(r) = \frac{(2-r^2)\varepsilon - 2(r')^2\kappa}{r^2(\kappa - \varepsilon)} \text{ and } h_7(r) = \frac{\varepsilon'}{r^2\kappa'}.$$

By [14, Theorem 14], h_6 is decreasing from $(0, 1)$ onto $(0, 3/4)$, while $h_7(r) = (r^{-1/2}\varepsilon')\cdot(r^{3/2}\kappa')^{-1}$ is strictly decreasing from $(0,1)$ onto $(1,\infty)$ by [6, Lemma 5.4]. Hence, it follows from (32) that

$$h_5(r) > H_3(a,b) \equiv h_6(b) - h_7(a) + \frac{3}{5} \quad (33)$$

for each $[a, b] \subset (0,1)$. Computation gives: $H_3(0.955, 0.975) = 0.0199\ldots$, $H_3(0.975, 0.982) = 0.0166\ldots$, $H_3(0.982, 0.987) = 0.0005\ldots$. Thus, by (27), (32) and (33), h_1 is strictly decreasing on $[0.955, 0.987]$, so that

$$h_1(r) \geqslant h_1(0.987) = [2\pi(\log 4)/5] - g_1(0.987) = 0.0285\ldots, \text{ for } r \in [0.955, 0.987] \quad (34)$$

It follows from (20), (27), (31) and (34) that h is strictly increasing on $[0.54, 1)$. This, together with the monotonicity of h obtained in step 1, yields

$$h(r) < h(0^+) = h(1^-) = 0, \text{ for } r \in (0, 0.53] \cup [0.54, 1) \quad (35)$$

Step 3 We prove that $h(r) < 0$ for $r \in [0.53, 0.54]$.

For $r \in [0.53, 0.54]$, we have

$h(r) \leqslant M(0.53) - (1 - 0.54^2)^{4/5} \log 4 = -0.0464\ldots$,

sinc $M(r)$ is decreasing on $(0,1)$ by [6, Theorem 5.5(3)]. This, together with (35), yields (26) as desired.

Next, we prove the second inequality in (11). Clearly,

$$M(r)\leqslant (r')^{\alpha}\log 4 \Leftrightarrow \alpha \leqslant \inf_{0<r<1} F(r),$$

where

$$F(r)=\frac{\log[(\log 4)/M(r)]}{\log(1/r')}.$$

Since $F(0.98)=1.6245\ldots$, we obtain the upper bound of α in (11).

Now let $h_8(r)=M(r)-(1-r)^{\beta}\log 4$ for $r\in(0,1)$, where $0<\beta\leqslant 1$. Then

$$\frac{\pi}{4}(1-r)^{1-\beta}h'_8(r)=h_9(r)\equiv$$

$$\frac{\pi}{2}\beta\log 2-(1-r)^{1-\beta}\kappa'\frac{\kappa-\varepsilon}{r} \tag{36}$$

If $\beta=1$, then $h_9(r)=[(\pi\log 2)/2]-r\kappa'\cdot(\kappa-\varepsilon)/r^2$ is strictly decreasing from $(0,1)$ onto $(-\infty,(\pi\log 2)/2)$ by [6, Lemmas 5.2(3) & 5.4(1)] and has a unique zero r_1 on $(0,1)$. Thus $h_8(r)$ is strictly increasing on $(0, r_1]$, and decreasing on $[r_1, 1)$. Hence $h_8(r)\geqslant 0$ for all $r\in(0,1)$. This yields the estimate: $\beta\leqslant 1$.

If $0<\beta<1$, then $h_9(1^-)=(\pi\beta\log 2)/2>0$, so that there exists a number $r_2\in(0,1)$ such that h_8 is strictly increasing on $[r_2, 1)$ and $h_8(r)<h_8(1^-)=0$ for $r\in[r_2,1)$. Hence $\beta\not<1$. Thus $\beta=1$.

The remaining conclusion in part (1) is clear.

(2) Part (2) follows from part (1) and (4).

(3) Part (3) follows from part (3) and (13).

Corollary 3.1　(1) There exists a unique number $r_0\in(0.53, 0.54)$ such that the function

$$h(r)\equiv M(r)-(r')^{8/5}\log 4$$

is strictly decreasing on $(0, r_0]$, and strictly increasing on $[r_0, 1)$, with $h(0^+)=h(1^-)=0$.

(2) There exists a unique number $r_1\in(0.59, 0.591)$ such that the function

$$h_8(r)\equiv M(r)-(1-r)\log 4$$

is strictly increasing on $(0, r_1]$, and strictly decreasing on $[r_1, 1)$, with $h_8(0^+)=h_8(1^-)=0$.

Proof　(1) We use the same notation as in the proof of Theorem 1.4. It follows from (32) that

$$h_5(r)<H_4(a,b)\equiv h_6(a)-h_7(b)+0.6 \tag{37}$$

for each $[a, b]\subset[0.53, 0.54]$. Since $H_4(0.53, 0.54)=-0.6860\ldots$, it follows from (27), (32) and (37) that h_1 is strictly increasing on $[0.53, 0.54]$. Since $h_1(0.53)=-0.0044\ldots$ and since $h_1(0.54)=0.0051\ldots$, h_1 has a unique zero r_0 on $[0.53, 0.54]$ such that $h_1(r)<0$ for $r\in[0.53, r_0)$, and $h_1(r)>0$ for $r\in(r_0, 0.54]$. Hence by (27), h is strictly decreasing on $[0.53, r_0]$, and strictly increasing on $[r_0, 0.54]$. This, together with the monotonicity of h on $(0, 0.53]$ and on $[0.54, 1)$ proved above, yields the piecewise monotonicity of h on $(0, 1)$.

(2) Let h_9 be as in (36) with $\beta=1$. Then by the proof of Theorem 1.4, h_9 is strictly decreasing from $(0,1)$ onto $(-\infty,(\pi\log 2)/2)$ and has a unique zero r_1 on $(0,1)$. Since $h_9(0.59)=0.00006\ldots$ and $h_9(0.591)=-0.00168\ldots$, $r_1\in(0.59, 0.591)$ and the conclusion follows.

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关于 φ_K-偏差函数的一类上界的最佳情形

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摘　要：研究了拟共形理论中著名的 Hübner 不等式中的函数 $M(r)=\frac{2}{\pi}(r')^2\kappa(r)\kappa'(r)+\log r$ 的形如 $(r')_\alpha\cdot\log 4$ 的上界估计中的最佳指数 α 为何值这一问题，获得了 $\max\{c$：不等式 $M(r)<(r')^c\log 4$ 对一切 $r\in(0,1)$ 成立$\}$ 的上下界估值，证明了 $\min\{c:M(r)>(1-r)^c\log 4$ 一切 $r\in(0,1)$ 成立$\}=1$。从而改进了已知的此类估计与由此类估计得出的拟共形理论中极为重要的 Hersch-Pfluger 偏差函数 $\varphi_K(r)$ 的上界以及相应的显式拟共形 Schwarz 引理。

关键词：上界估计；Hübner 不等式；Hersch-Pfluger 偏差函数；拟共形 Schwarz 引理

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