浙江理工大学学报，第 28 卷，第 5 期，2011 年 9 月
Journal of Zhejiang Sci-Tech University
Vol. 28，No. 5，Sept. 2011

文章编号：1673-3851 (2011) 05-0819-06

一类多元 Szasz-Mirakjan 算子线性组合的一致逼近

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摘　要：运用 K-泛函方法，主要研究一类多元 Szasz-mirakjan 算子线性组合的逼近问题，建立一致逼近下的正、逆定理，并给出逼近阶的估计和特征刻划。从而将一元 Szasz-Mirakjan 算子及其线性组合的相关结果推广到多元情形。

关键词：多元 Szasz-Mirakjan 算子；线性组合；一致逼近；特征刻划
中图分类号：O174.41　　**文献标识码**：A

0　引　言

一元 Szasz-Mirakjan 算子定义为：

$$S_n(f;x)=\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)P_{n,k}(x),P_{n,k}(x)=\mathrm{e}^{-nx}\frac{(nx)^k}{k!},x\geqslant 0,n\in N. \tag{1}$$

一元 Szasz 型算子线性组合的点态逼近问题[1]，以及多元 Szasz-Mirakjan 算子的一致逼近问题[2]，已有较为深入的研究，并取得相应的逼近结论。然而关于该算子多元情形的线性组合的研究却很少。

本文旨在作这方面的尝试。为方便计，仅限于讨论二元情形，其余多元情形可类似讨论。文中将二元 Szasz-Mirakjan 算子 $S_{n,m}(f;x)$构造为

$$S_{n,m}(f;x,y)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}f\left(\frac{k}{n},\frac{l}{m}\right)P_{n,k}(x)P_{m,l}(y) \tag{2}$$

其线性组合可“形式”定义为：

$$S_{n,m}(f,r;x,y)=\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)\cdot S_{n_i,m_j}(f;x,y)\quad r>1 \tag{3}$$

其中 $n_i,m_j\in N$，且 n_i,m_j 及 $C_{i,j}(n,m)$满足如下条件(k,C,k_1,k_2 为绝对常数)

$$\text{(a)}\begin{cases}n=n_0<n_1\cdots<n_{r-1}\leqslant kn\\ m=m_0<m_1\cdots<m_{r-1}\leqslant km\end{cases}\qquad \text{(b)}\sum_{i=0}^{r-1}\sum_{j=0}^{i}|C_{i,j}(n,m)|\leqslant C$$

$$\text{(c)}\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)=1\qquad \text{(d)}\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)n_i^{-\rho}m_j^{\rho-\sigma}=0 \tag{4}$$

$$\text{(e)}0<k_1\leqslant\frac{n}{m}\leqslant k_2<+\infty,\rho=0,1,2,\cdots\sigma;\sigma=1.2,\cdots r-1$$

如上定义的线性组合是唯一存在的[3]。

本文将建立这类二元 Szasz-Mirakjan 算子线性组合逼近的正逆定理，从而给出如下等价关系

$$\|S_{n,m}(f,r;x,y)-f(x,y)\|_{\infty}=O(n^{-\frac{\alpha}{2}})\Leftrightarrow\omega_{\varphi}^{2r}(f,h)=O(h^{\alpha}) \tag{5}$$

这里设 $f\in C(R^2)\cap L_{+\infty}(R^2)$，$0<\alpha<2r$，而

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$$\| f \| = \sup_{(x,y)\in R^2} |f(x,y)|, \omega_\varphi^{2r}(f,t) = \sup_{0<h\leqslant t} \sup_{\max|v_i|=1} \| \Delta_{h,\varphi}^r f(x,y) \|$$

$$\Delta_{h,\varphi}^r f(x,y) = \sum_{k=0}^{r} \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right)h\varphi v_1, y + \left(\frac{r}{2} - k\right)h\varphi v_2\right)$$

如下关系成立[4]：

$$\omega_\varphi^{2r}(f,t) \sim K_{2r,\varphi}(f,t^{2r}) \tag{6}$$

这里

$$K_{2r,\varphi}(f,t^{2r}) = \inf_{g\in D}\{ \| f-g \| + t^{2r}S(g)\}, S(g) = \max_{0\leqslant k\leqslant 2r} \left\| \varphi^{2r} \cdot \frac{\partial^{2r} g}{\partial x^k \partial y^{2r-k}} \right\|, \varphi(x) = \sqrt{x}$$

D 为如下的 Sobolev 空间：

$$D = \{ f | f \in C(R_+^2) \cap L_{+\infty}(R_+^2)\}, \frac{\partial^{2r-1} f(x,y)}{\partial x^k \partial y^{2r-1-k}} \in A \cdot C_{loc}, 0 \leqslant k \leqslant 2r-1, S(f) < +\infty\}$$

1 引 理

为证明本文的主要结论，需要先证明下列引理：

引理 1.1 设 $f \in C(R_+^2) \cap L_{+\infty}(R_+^2)$，则有

$$\| S_{n,m}(f,r;x,y) \| \leqslant M \| f \| \tag{7}$$

这里 M_r 为仅与 r 有关而与 f,n,m 无关的常数，M 为绝对常数(下同)。

证明 由文献[2]引理 1 知：$\| S_{n_i,m_j}(f;x,y) \| \leqslant M \| f \|$，又由式(3)和式(4)(b)可得：

$$| S_{n,m}(f,r;x,y) | = \left| \sum_{i=0}^{r-1} \sum_{j=0}^{i} C_{i,j}(n,m) \cdot S_{n_i,m_j}(f;x,y) \right| \leqslant$$

$$\| S_{n_i,m_j}(f;x,y) \| \cdot \sum_{i=0}^{r-1} \sum_{j=0}^{i} | C_{i,j}(n,m) | \leqslant M \| f \|$$

从而引理 1.1 得证。

引理 1.2 设 $f \in C(R_+^2) \cap L_{+\infty}(R_+^2)$，则有

$$\| N(S_{n,m}(f,r;x,y)) \| \leqslant M_r \cdot n^r \| f \| \tag{8}$$

对 $f \in D$，记 $N(f) = \sum_{k=0}^{2r} \left\| \varphi^k(x) \varphi^{2r-k}(y) \cdot \frac{\partial^{2r} f}{\partial x^k \partial y^{2r-k}} \right\|$.

证明 当 $x > \frac{1}{n}$ 时

$$\left\| \frac{\partial^{2r}}{\partial x^s \partial y^{2r-s}} S_{n_i,m_j}(f;x,y) \right\| = \left| \frac{\partial^{2r}}{\partial x^s \partial y^{2r-s}} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} f\left(\frac{k}{n_i}, \frac{l}{m_j}\right) P_{n_i,k}(x) P_{n_j,l}(y) \right| =$$

$$\left| \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} f\left(\frac{k}{n_i}, \frac{l}{m_j}\right) P_{n_i,k}^{(s)}(x) P_{m_j,l}^{(2r-s)}(y) \right| \leqslant$$

$$\| f \| \sum_{k=0}^{\infty} P_{n_i,k}^{(s)}(x) \Big| \cdot \sum_{l=0}^{\infty} | P_{m_i,l}^{(2r-s)}(y) \Big|$$

由关系式[1]，$P_{n_i,k}^{(s)} = \sum_{2a+b\leqslant s} q_{a,b,s}(k - n_i x)^{s-2a-b} x^{a-s} n_i^a P_{n_i,k}(x)$

其中 $l \leqslant 0, m \leqslant 0$ 且为整数，$q_{a,b,s}$ 为与 k 及 x 无关，关于 n_i 一致有界的常数。

$$\sum_{k=0}^{\infty} | \varphi^s(x) P_{n_i,k}^{(s)}(x) | \leqslant C \sum_{k=0}^{\infty} \sum_{2a+b\leqslant s} | k - n_i x |^{s-2a-b} x^{a-\frac{s}{2}} n_i^a P_{n_i,k}(x) =$$

$$C \sum_{2a+b\leqslant s} x^{a-\frac{s}{2}} n_i^a \sum_{k=0}^{\infty} | k - n_i x |^{s-2a-b} P_{n_i,k}(x) =$$

$$C \sum_{2a+b\leqslant s} x^{a-\frac{s}{2}} S_{n_i}(| t - x |^{s-2a-b}; x)$$

注意到文献[4] $S_n((t-x)^{2m}; x) = \sum_{i=0}^{m-1} q_{i,m} \left(\frac{x}{n_i}\right)^{m-i} n_i^{-2i}, m \in N$

此处 $q_{i,m}$ 与 x 无关，关于 n_i 一致有界。所以 $S_n((t-x)^{2m};x)\leqslant C\left(\frac{x}{n}\right)^m$。

于是由 Cauchy-Schwartz 不等式

$$\sum_{k=0}^{\infty}|\varphi^s(x)P_{n_i,k}^{(s)}(x)|\leqslant C\sum_{2a+b\leqslant s}x^{a-\frac{s}{2}}S_{n_i}(|t-x|^{2s-4a-2b};x)^{\frac{1}{2}}\leqslant$$
$$C\sum_{2a+b\leqslant s}(n_ix)^{-\frac{b}{2}}n_i^{\frac{s}{2}}\leqslant C\cdot n_i^{\frac{s}{2}}$$

同理可证

$$\sum_{k=0}^{\infty}|\varphi^{2r-s}(y)P_{m_j,k}^{(2r-s)}(y)|\leqslant C\cdot m_j^{\frac{2r-s}{2}}$$

所以有 $\left|\varphi^{2r}\cdot\frac{\partial^{2r}}{\partial x^s\partial y^{2r-s}}S_{n_i,m_j}(f;x,y)\right|\leqslant M_rm_j^{\frac{2r-s}{2}}n_i^{\frac{s}{2}}\|f\|$

当 $0\leqslant x<\frac{1}{n}$ 时，利用 Z. DitzianV. Totik 文献[4]中相同的方法可以证明

$$\left|\varphi^{2r}\cdot\frac{\partial^{2r}}{\partial x^s\partial y^{2r-s}}S_{n_i,m_j}(f;x,y)\right|\leqslant M_rn_i^r\|f\|$$

从而

$$\left|\varphi^{2r}\cdot\frac{\partial^{2r}}{\partial x^k\partial y^{2r-k}}S_{n_i,m_j}(f;x,y)\right|\leqslant M_rn^r\|f\|$$

故引理结论成立。

引理 1.3　设 $f\in D$，则有

$$\|N(S_{n,m}(f,r;x,y))\|\leqslant M_r\cdot N(f)\tag{9}$$

证明　利用 Taylor 公式：

$$f(u;v)=\sum_{h=0}^{2r-1}\frac{1}{h!}\left[(u-x)\frac{\partial}{\partial x}+(v-y)\frac{\partial}{\partial y}\right]^hf(x,y)+R_{2r}(f,x,y,u,v)\tag{10}$$

其中

$$R_{2r}(f,x,y,u,v)=\frac{1}{(2r-1)!}\int_0^1\left[(u-x)\frac{\partial}{\partial x}+(v-y)\frac{\partial}{\partial y}\right]^{2r}$$
$$f([x+t(u-x),y+t(v-y)](1-t)^{2r-1}\mathrm{d}t$$

从而得 $\varphi^s(x)\cdot\varphi^{2r-s}(y)\frac{\partial^{2r}}{\partial x^s\partial y^{2r-s}}S_{n_i,m_j}(f;x,y)=$

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{h=0}^{2r-1}\frac{1}{h!}\left[\left(\frac{k}{n_i}-x\right)\frac{\partial}{\partial x}+\left(\frac{l}{m_j}-y\right)\frac{\partial}{\partial y}\right]^hf(x,y)P_{n_i,k}^{(s)}(x)P_{m_j,l}^{(2r-s)}(y)\varphi^s(x)\varphi^{2r-s}(y)+$$
$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}R_{2r}(f,x,y,u,v)P_{n_i,k}^{(s)}(x)P_{m_j,l}^{(2r-s)}(y)\varphi^s(x)\varphi^{2r-s}(y)=:I+J$$

$$I=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{h=0}^{2r-1}\frac{1}{h!}\sum_{\tau=0}^{h}\binom{h}{\tau}\frac{\partial^hf(x,y)}{\partial x^\tau\partial y^{h-\tau}}\left(\frac{k}{n_i}-x\right)^\tau\left(\frac{l}{m_j}-y\right)^{h-\tau}P_{n_i,k}^{(s)}(x)P_{m_j,l}^{(2r-s)}(y)\varphi^s(x)\varphi^{2r-s}(y)=$$
$$\sum_{h=0}^{2r-1}\frac{1}{h!}\sum_{\tau=0}^{h}\binom{h}{\tau}\frac{\partial^hf(x,y)}{\partial x^\tau\partial y^{h-\tau}}\sum_{k=0}^{\infty}\left(\frac{k}{n_i}-x\right)^\tau P_{n_i,k}^{(s)}(x)\varphi^s(x)\sum_{l=0}^{\infty}\left(\frac{l}{m_j}-y\right)^{h-\tau}P_{m_j,l}^{(2r-s)}(y)\varphi^{2r-s}(y)$$

因为 $\tau<s$ 与 $h-\tau<2r-s$ 必有一式成立

所以 $\sum_{k=0}^{\infty}\left(\frac{k}{n_i}-x\right)^\tau P_{n_i,k}^{(s)}(x)\varphi^s(x)=0$ 与 $\sum_{l=0}^{\infty}\left(\frac{l}{m_j}-y\right)^{h-\tau}P_{m_j,l}^{(2r-s)}(y)\varphi^{2r-s}(y)=0$

必有一式成立，即 $I=0$。

$$|J|=\left|\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{(2r-1)!}\int_0^1\left[\left(\frac{k}{n_i}-x\right)\frac{\partial}{\partial x}+\left(\frac{l}{m_j}-y\right)\frac{\partial}{\partial y}\right]^{2r}f\left[x+t\left(\frac{k}{n_i}-x\right),y+t\left(\frac{l}{m_j}-y\right)\right]\right.$$
$$\left.(1-t)^{2r-1}\mathrm{d}tP_{n_i,k}^{(s)}(x)\varphi^s(x)P_{m_j,l}^{(2r-s)}(y)\varphi^{2r-s}(y)\right|=$$

$$\left|\frac{1}{(2r-1)!}\sum_{\tau=0}^{2r}\binom{2r}{\tau}\int_0^1\frac{\partial^{2r}f}{\partial x^\tau\partial y^{2r-\tau}}(1-t)^{2r-1}\mathrm{d}t\sum_{k=0}^{\infty}\left(\frac{k}{n_i}-x\right)^\tau\right.$$

$$\left.P_{n_i,k}^{(s)}(x)\varphi^s(x)\sum_{l=0}^{\infty}\left(\frac{l}{m_j}-y\right)^{2r-\tau}P_{m_j,l}^{(2r-s)}(y)\varphi^{2r-s}(y)\right|\leqslant$$

$$C\sum_{l=0}^{2r}\left\|\frac{\partial^{2r}f}{\partial x^\tau\partial y^{2r-\tau}}\right\|\sum_{\tau=0}^{2r}\left|\sum_{k=0}^{\infty}\left(\frac{k}{n_i}-x\right)^\tau P_{n_i,k}^{(s)}(x)\varphi^s(x)\sum_{l=0}^{\infty}\left(\frac{l}{m_j}-y\right)^{2r-\tau}P_{m_j,l}^{(2r-s)}(y)\varphi^{2r-s}(y)\right|$$

设 $A_{i,j}(n,x)=\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^j P_{n,k}^{(i)}(x)\varphi^i(x)$。

则当 $x>\frac{1}{n}$ 时，$S_{n,k}^{(i)}=\sum_{2a+b\leqslant i}q_{a,b,i}(k-nx)^{i-2a-b}x^{a-i}n^a P_{n,k}(x)$，

其中 $l\geqslant0,m\geqslant0$ 且为整数，$q_{a,b,s}$ 与 k 及 x 无关，关于 n 一致有界。

从而

$$A_{i,j}(n,x)=\sum_{k=0}^{\infty}\sum_{2a+b\leqslant i}q_{a,b,i}(k-nx)^{i-2a-b}x^{a-i}n^a\left(\frac{k}{n}-x\right)^j\varphi^i(x)P_{n,k}(x)\leqslant$$

$$C\sum_{2a+b\leqslant i}x^{a-\frac{i}{2}}n^{i-a-b}\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{i+j-2a-b}P_{n,k}(x)$$

考虑到 $\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{i+j-2a-b}P_{n,k}(x)\leqslant C\left(\frac{x}{n}\right)^{\frac{(i+j-2a-b)}{2}}$，则 $A_{i,j}(n,x)\leqslant Cx^{\frac{j}{2}}n^{\frac{i-j}{2}}$

从而可得

$$\sum_{k=0}^{\infty}\left(\frac{k}{n_i}-x\right)^\tau P_{n_i,k}^{(s)}(x)\varphi^s(x)\leqslant c\varphi^\tau(x)n_i^{\frac{s-r}{2}}$$

$$\sum_{l=0}^{\infty}\left(\frac{l}{m_j}-y\right)^{2r-\tau}P_{n_i,k}^{(2r-S)}(x)\varphi^{2r-s}(x)\leqslant c\varphi^{2r-\tau}(y)m_j^{\frac{\tau-s}{2}}$$

所以有 $|J|\leqslant C\left\|\frac{\partial^{2r}f(x,y)}{\partial x^\tau\partial y^{2r-\tau}}\right\|\sum_{\tau=0}^{2r}\varphi^\tau(x)n_i^{\frac{s-\tau}{2}}\varphi^{2r-\tau}(y)m_j^{\frac{\tau-s}{2}}$

即 $|J|\leqslant C\sum_{\tau=0}^{2r}\left\|\frac{\partial^{2r}f(x,y)}{\partial x^\tau\partial y^{2r-\tau}}\varphi^\tau(x)\varphi^{2r-\tau}(y)\right\|\leqslant M_r\cdot N(f)$。

故式(9)成立。

引理 1.4 设 $f\in D$，则有

$$\|S_{n,m}(f,r;x,y)-f(x,y)\|\leqslant M_r\cdot n^{-r}\cdot N(f)\tag{11}$$

证明 由式(3)得

$$S_{n,m}(f,r;x,y)-f(x,y)=\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)[S_{n_i,m_j}(f;x,y)-f(x,y)]=$$

$$\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{h=1}^{2r-1}\frac{1}{h!}\left[\left(\frac{k}{n_i}-x\right)\frac{\partial}{\partial x}+\left(\frac{l}{m_j}-y\right)\frac{\partial}{\partial y}\right]^h$$

$$f(x,y)P_{n_i,k}(x)P_{m_j,l}(y)+\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}R_{2r}$$

$$(f,x,y,u,v)P_{n_i,k}(x)P_{m_j,l}(y)=:I_1+J_1$$

其中由式(7)和文献[3]注1.3得

$$|I_1|=\left|\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)\cdot\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{h=1}^{2r-1}\frac{1}{h!}\sum_{\tau=0}^{h}\binom{h}{\tau}\frac{\partial^h f(x,y)}{\partial x^\tau\partial y^{h-\tau}}\left(\frac{k}{n_i}-x\right)^\tau\left(\frac{l}{m_j}-y\right)^{2r-\tau}\right.$$

$$\left.P_{n_i,k}(x)P_{m_j,l}(y)\right|=\left|\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)\cdot\sum_{h=1}^{2r-1}\frac{1}{h!}\sum_{\tau=0}^{h}\binom{h}{\tau}\frac{\partial^h f(x,y)}{\partial x^\tau\partial y^{h-\tau}}\sum_{k=0}^{\infty}\left(\frac{k}{n_i}-x\right)^\tau\right.$$

$$\left.P_{n_i,k}(x)\sum_{l=0}^{\infty}\left(\frac{l}{m_j}-y\right)^{h-\tau}P_{m_j,l}(y)\right|\leqslant C\left|\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)\cdot\sum_{h=1}^{2r-1}\frac{1}{h!}\sum_{\tau=0}^{h}\binom{h}{\tau}\frac{\partial^h f(x,y)}{\partial x^\tau\partial y^{h-\tau}}\right.$$

$$\varphi^{\tau}(x)n_i^{\frac{-\tau}{2}}\varphi^{h-\tau}(y)m_j^{\frac{r}{2}-\frac{h}{2}}\Big| \leqslant C\Big|\sum_{h=1}^{2r-1}\frac{1}{h!}\sum_{\tau=0}^{h}\binom{h}{\tau}\frac{\partial^h f(x,y)}{\partial x^{\tau}\partial y^{h-\tau}}\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)n_i^{\frac{-r}{2}}m_j^{\frac{r}{2}-\frac{h}{2}}\varphi^{\tau}(x)\varphi^{h-\tau}(y)\Big| \leqslant$$

$$C\Big|\sum_{h=1}^{2r-1}\frac{1}{h!}\sum_{\tau=0}^{h}\binom{h}{\tau}\frac{\partial^h f(x,y)}{\partial x^{\tau}\partial y^{h-\tau}}\varphi^{\tau}(x)\varphi^{h-\tau}(y)\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)n_i^{\frac{-r}{2}}m_j^{\frac{r}{2}-\frac{h}{2}}\Big| = 0$$

$$|J_1| = \Big|\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)\cdot\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}R_{2r}(f,x,y,u,v)P_{n_i,k}(x)P_{m_j,l}(y)\Big| = \Big|\sum_{i=0}^{r-1}\sum_{j=0}^{i}C_{i,j}(n,m)\cdot$$

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\Big\{\frac{1}{(2r-1)!}\int_0^1\Big[\Big(\frac{k}{n_i}-x\Big)\frac{\partial}{\partial x}+\Big(\frac{l}{m_j}-y\Big)\frac{\partial}{\partial y}\Big]^{2r}\cdot f\Big[x+t\Big(\frac{k}{n_i}-x\Big),$$

$$y+t\Big(\frac{l}{m_j}-y\Big)\Big]\cdot(1-t)^{2r-1}\mathrm{d}t\Big\}P_{n_i,k}(x)P_{m_j,l}(y) \leqslant C_r\sum_{i=0}^{r-1}\sum_{j=0}^{i}|C_{i,j}(n,m)|$$

$$\Big(\sum_{\tau=0}^{2r}\Big\|\varphi^{\tau}(x)\cdot\varphi^{2r-\tau}(y)\frac{\partial^{2r}f}{\partial x^{\tau}\partial y^{2r-\tau}}\Big\|\cdot\sum_{k=0}^{\infty}\Big|\Big(\frac{k}{n_i}-x\Big)^{\tau}\varphi^{-\tau}(x)P_{n_i,k}(x)\Big|\sum_{l=0}^{\infty}\Big|\Big(\frac{l}{m_j}-y\Big)^{2r-\tau}$$

$$\varphi^{\tau-2r}(y)P_{m_j,l}(y)\Big|\Big)\ \|J_1\| \leqslant C_r\cdot n_i^{\frac{-\tau}{2}}\cdot m_j^{\frac{\tau-2r}{2}}\sum_{\tau=0}^{2r}\Big\|\varphi^{\tau}(x)\cdot\varphi^{2r-\tau}(y)\frac{\partial^{2r}f}{\partial x^{\tau}\partial y^{2r-\tau}}\Big\|$$

从而

$$\|S_{n,m}(f,r;x,y)-f(x,y)\| \leqslant M_r\cdot n^{-r}N(f)$$

引理 1.4 成立。

2 主要结果的证明

定理 2.1(逼近正定理),设 $f\in C(R^2)$,则有

$$\|S_{n,m}(f,r;x,y)-f(x,y)\| \leqslant M_r\omega_{\varphi}^{2r}\Big(f,n^{-\frac{1}{2}}\Big) \tag{12}$$

证明 由定义知 $N(f)\sim S(f)$,取 $g\in D$,则由引理 1.1 和引理 1.4 可得

$$\|S_{n,m}(f,r;x,y)-f(x,y)\| \leqslant \|S_{n,m}(f-g,r;x,y)-(f-g)\|+\|f-g\|+\|S_{n,m}(g,r;x,y)-g(x,y)\| \leqslant M_r(\|f-g\|+n^{-r}S(g))$$

上式两边对 g 取下确界,即得

$$\|S_{n,m}(f,r;x,y)-f(x,y)\| \leqslant M_r\cdot K_{2r,\varphi}\Big(f,\Big(n^{-\frac{1}{2}}\Big)^{2r}\Big)\sim M_r\omega_{\varphi}^{2r}\Big(f,n^{-\frac{1}{2}}\Big)$$

定理 2.2(逼近逆定理),设 $f\in C(R^2)$,则有

$$K_{2r,\varphi}(f,n^{-r}) \leqslant \|S_{k_1,k_2}(f,r;x,y)\|+M_r\Big(\frac{k_1}{n}\Big)^{r}K_{2r,\varphi}(f,k_1^{-r}) \tag{13}$$

证明 首先由 K-泛函的定义有

$$K_{2r,\varphi}(f,n^{-r}) \leqslant \|f-S_{k_1,k_2}(f,r;x,y)\|+n^{-r}S(S_{k_1,k_2}(f,r;x,y)) \tag{14}$$

这里 $k_1\leqslant n,k_2\leqslant m$,取 $g\in D$,则由引理 1.2 和引理 1.3 得

$$S(S_{k_1,k_2}(f-g,r;x,y))\sim N(S_{k_1,k_2}(f-g,r;x,y)) \leqslant M_r\cdot k_1^r\|f-g\|$$

$$S(S_{k_1,k_2}(g,r;x,y))\sim N(S_{k_1,k_2}(g,r;x,y) \leqslant M_r\cdot N(g)$$

从而得

$$n^{-r}S(S_{k_1,k_2}(f,r;x,y)) \leqslant M_r\cdot n^{-r}(k_1^r\|f-g\|+S(g)) \leqslant M_r\Big(\frac{k_1}{n}\Big)^{r}(\|f-g\|+k_1^{-r}S(g))$$

上式两边对 g 取下确界,即得

$$n^{-r}S(S_{k_1,k_2}(f,r;x,y) \leqslant M_r\Big(\frac{k_1}{n}\Big)^{r}K_{2r,\varphi}(f,k_1^{-r}) \tag{15}$$

将式(15)代入式(14)即知式(13)成立。

定理 2.3(逼近等价定理) 设 $f\in C(R^2)$,$0<\alpha<2r$。则有

$$\|S_{n,m}(f,r;x,y)-f(x,y)\| = O\Big(n^{\frac{-\alpha}{2}}\Big) \Leftrightarrow \omega_{\varphi}^{2r}(f,h) = O(h^{\alpha}) \tag{16}$$

证明 由定理2.1和定理2.2及Berebs-Lorentz引理[4]知式(16)成立。

定理2.3证毕。

衷心感谢宣培才教授、王建力教授的悉心指导和仔细审阅原稿,感谢审稿人对本文修改提出宝贵意见!

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Uniform Approximation by Class of Combinations of Multidimensional Szasz-Mirakjan Operators

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Abstract:In this paper,a class of linear combinations of multidimensional Szasz-Mirakjan operators are considered by K-factional method. A theorem and its reverse under uniform approximation are established,the estimate and characterization of the order of the approximation are alse given. Then,the anthor extend the results of one-dimensional Szasz-Mirakjan operators to multidimension results.

Key words:multidimensional szasz-mirakjan operators;linear combination;uniform approximation;characterization

(责任编辑:马春晓)

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