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Comparisons Between the Ramanujan Constant and Beta Function

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Abstract: In this paper, the authors show the monotonicity and convexity properties of certain combinations defined in terms of the Ramanujan constant $R(a)=-2\gamma-\psi(a)-\psi(1-a)$ and beta function $\mathrm{B}(a)=\mathrm{B}(a,1-a)=\pi/\sin(\pi a)$, and obtain asymptotically sharp lower and upper bounds for $R(a)-\mathrm{B}(a)$. Thus, they deeply reveal the relationship between $R(a)$ and $\mathrm{B}(a,1-a)$ and improve some known results for $R(a)$.

Key words: Ramanujan constant; psi and beta functions; monotonicity; convexity; lower and upper bounds

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0 Introduction and Main Results

For $x, y>0$, the gamma, beta and psi functions are defined as

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t, \mathrm{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \psi(x)=\frac{\Gamma'(x)}{\Gamma(x)} \tag{1}$$

respectively. For their basic properties, the reader is referred to [1-4]. Here we only record following formulas applied frequently in the sequel, (cf. [1, 6.3.5, 6.3.7, 6.3.16, 6.4.10, 6.3.22 & 6.1.17], respectively):

$$\psi(1-x)-\psi(x)=\pi\cot(\pi x), \tag{2}$$

$$\psi(x)+\frac{1}{x}=\psi(1+x)=-\gamma+\sum_{n=1}^\infty\frac{x}{n(n+x)}, \tag{3}$$

$$\psi^{(n)}(x)=(-1)^{n+1}n!\sum_{k=0}^\infty\frac{1}{(k+x)^{n+1}}, \tag{4}$$

$$\psi(x)+\gamma=\int_0^1\frac{1-t^{x-1}}{1-t}\mathrm{d}t, \frac{\pi}{\sin(\pi x)}=\int_0^\infty\frac{t^{x-1}}{1+t}\mathrm{d}t. \tag{5}$$

The so-called Ramanujan constant $R(a)$ is defined by

$$R(a)\equiv-2\gamma-\psi(a)-\psi(1-a). \tag{6}$$

for $a\in(0,1)$, where $\gamma=0.5772156649\cdots$ is the Euler constant. By the symmetry, we can assume that $a\in(0,1/2]$ in (6). It is well known that $R(a)$ is essential not only in the study of the generalized elliptic integrals and the theory of Ramanujan's modular equations, but also in some other fields of mathematics such as quasiconformal theory. (See [1 − 3 & 5 − 11].) Some authors have obtained various analytic properties and functional inequalities for this function. (Cf. [3, 5 & 8-13].) On the other hand, the

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functions $R(a)$ and

$$\mathrm{B}(a) \equiv \mathrm{B}(a, 1-a) = \Gamma(a)\Gamma(1-a) = \pi/\sin(\pi a) \tag{7}$$

often simultaneously appear in the related studies, and we have to reveal the relations between $R(a)$ and $\mathrm{B}(a)$.

The main purpose of this paper is to show the monotonicity and convexity properties of certain combinations defined in terms of $R(a)$, $\mathrm{B}(a)$ and some elementary functions, and to obtain asymptotically sharp lower and upper bounds for $R(a) - \mathrm{B}(a)$. By these results, several comparisons between $R(a)$ and $\mathrm{B}(a)$ are presented, and some known related results are improved.

Throughout this paper, we let

$$\alpha = \frac{\log 16}{\pi} = 0.8825\cdots, \beta = 2\pi - 8\log 2 = 0.7380\cdots, \delta = \frac{\pi^2}{6} = 1.6449..., \eta = \delta - \beta = 0.9069\cdots,$$

and let $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ denote the Riemann zeta function as usual, with Re $s > 1$. We now state the main results of this paper below.

Theorem 1 a) The function $f(x) \equiv R(x)/\mathrm{B}(x)$ is strictly decreasing from $(0, 1/2]$ onto $[\alpha, 1)$. However, f is neither convex nor concave on $(0, 1/2]$.

b) For $c \in (0, \infty)$, let $g(x) = R(x) - c\mathrm{B}(x)$. Then g is strictly decreasing on $(0, 1/2]$ if and only if $c \leqslant 1$, and g is convex on $(0, 1/2]$ if and only if $c \leqslant 1$, with $g((0, 1/2]) = [\log 16 - c\pi, \infty)$ if $c < 1$, and $g((0, 1/2]) = [-\beta/2, 0)$ if $c = 1$. Moreover, for $c \in (0, 1]$ and $n \in \mathbf{N}$, $g^{(2n)}$ is strictly decreasing and convex on $(0, 1/2]$, while $g^{(2n-1)}$ is strictly increasing and concave on $(0, 1/2]$. However, if $1 < c < 28\zeta(3)/\pi^3$, then g is not monotone on $(0, 1/2]$.

c) If $c = \alpha$, then the function $h(x) \equiv g(x)/(1-2x)^2$ is strictly decreasing and convex from $(0, 1/2)$ onto $([7\zeta(3) - \pi^2 \log 2]/2, \infty)$. In particular, for $x \in (0, 1/2]$,

$$R(x) \geqslant \alpha \mathrm{B}(x) + \frac{1}{2}[7\zeta(3) - \pi^2 \log 2](1-2x)^2, \tag{8}$$

with equality if and only if $x = 1/2$.

Theorem 2 a) The function $F(x) \equiv [\mathrm{B}(x) - R(x)]/x$ is strictly decreasing and convex from $(0, 1/2]$ onto $[\beta, \delta)$. In particular, for $x \in (0, 1/2]$,

$$\mathrm{B}(x) - \beta x - \eta x(1-2x) \leqslant R(x) \leqslant \mathrm{B}(x) - \beta x \tag{9}$$

with equality in each instance if and only if $x = 1/2$. Furthermore, the derivative F' is strictly increasing and concave from $(0, 1/2]$ onto $(-2\zeta(3), -2\beta]$.

b) The function $G(x) \equiv \frac{1}{x}\{\mathrm{B}(x) - R(x) + (1-\alpha)[\mathrm{B}(x) - (1/x)]\}$ is strictly decreasing and convex from $(0, 1/2]$ onto $[A_1, A_2)$, where $A_1 = 2\beta\left(1 - \frac{1}{\pi}\right) = 1.0061\cdots$ and $A_2 = \delta(2-\alpha) = 1.8381\cdots$. In particular, for $x \in (0, 1/2]$,

$$(2-\alpha)\mathrm{B}(x) - \frac{1-\alpha}{x} - A_1 x - (A_2 - A_1)x(1-2x) \leqslant R(x) \leqslant (2-\alpha)\mathrm{B}(x) - \frac{1-\alpha}{x} - A_1 x \tag{10}$$

with equality in each instance if and only if $x = 1/2$.

1 Proof of Theorem 1

In this section, we prove Theorem 1 stated in Section 0.

a) Set $y = x(1-x)$. It follows from (3) and (6) that

$$R(x) = \frac{1}{y} - 2\gamma - \psi(x+1) - \psi(2-x) = \frac{1}{y} - \sum_{k=1}^{\infty} \frac{k+2y}{k(k^2+k+y)}. \tag{11}$$

By (2)-(4), (6)-(7) and (11), and by differentiation, we obtain

$$\begin{aligned}\frac{\mathrm{B}(x)f'(x)}{1-2x} &= \frac{1}{1-2x}\{\psi'(1-x)-\psi'(x)+\mathrm{R}(x)[\psi(1-x)-\psi(x)]\}\\
&= \frac{1}{1-2x}\sum_{n=0}^{\infty}\left[\frac{1}{(n+1-x)^2}-\frac{1}{(n+x)^2}\right]+\\
&\quad \frac{1}{1-2x}\left[\frac{1}{y}-\sum_{n=1}^{\infty}\frac{n+2y}{n(n^2+n+y)}\right]\left[\frac{1-2x}{y}+\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1-x}{n+1-x}-\frac{x}{n+x}\right)\right]\\
&= -\sum_{n=0}^{\infty}\frac{2n+1}{(n^2+n+y)^2}+\frac{1}{y^2}\left[1-\sum_{n=1}^{\infty}\frac{y(n+2y)}{n(n^2+n+y)}\right]\left(1+\sum_{n=1}^{\infty}\frac{y}{n^2+n+y}\right)\\
&= -\sum_{n=1}^{\infty}\frac{2n+1}{(n^2+n+y)^2}+\\
&\quad \frac{1}{y^2}\left[\sum_{n=1}^{\infty}\frac{y}{n^2+n+y}-\sum_{n=1}^{\infty}\frac{y(n+2y)}{n(n^2+n+y)}-\left(\sum_{n=1}^{\infty}\frac{y}{n^2+n+y}\right)\sum_{n=1}^{\infty}\frac{y(n+2y)}{n(n^2+n+y)}\right]\\
&= -\sum_{n=1}^{\infty}\frac{2n+1}{(n^2+n+y)^2}-2\sum_{n=1}^{\infty}\frac{1}{n(n^2+n+y)}-\left(\sum_{n=1}^{\infty}\frac{1}{n^2+n+y}\right)\sum_{n=1}^{\infty}\frac{n+2y}{n(n^2+n+y)}<0.\end{aligned}$$

This yields the monotonicity of f.

Clearly, $f(1/2)=(\log 16)/\pi$. By (11), $f(0^+)=\lim_{x\to 0}[xR(x)][\sin(\pi x)]/(\pi x)=1$.

Next, differentiation gives

$$\pi f''(x)=[R''(x)-\pi^2R(x)+2\pi R'(x)\cot(\pi x)]\sin(\pi x). \tag{12}$$

By (3) and (6), we have

$$R(x)=-2\gamma-\psi(1-x)-\psi(1+x)+(1/x), \tag{13}$$

$$R'(x)=\psi'(1-x)-\psi'(1+x)-(1/x^2), \tag{14}$$

$$R''(x)=-\psi''(1-x)-\psi''(1+x)+(2/x^3). \tag{15}$$

By (12),

$$f''(x)=x[R''(x)-\pi^2R(x)+2\pi R'(x)\cot(\pi x)]\cdot\frac{\sin(\pi x)}{\pi x} \tag{16}$$

and hence it follows from (13)-(15) that

$$\begin{aligned}f''(0^+) &= \lim_{x\to 0}x\{-\psi''(1-x)-\psi''(1+x)+(2/x^3)+\pi^2[2\gamma+\psi(1-x)+\psi(1+x)-(1/x)]+\\
&\quad 2\pi[\psi'(1-x)-\psi'(1+x)-(1/x^2)]\cot(\pi x)\}\\
&= 2\lim_{x\to 0}\frac{1-\pi x\cot(\pi x)}{x^2}-\pi^2=2\lim_{x\to 0}\frac{\sin(\pi x)-\pi x\cos(\pi x)}{\pi x^3}-\pi^2=-\frac{\pi^2}{3}<0.\end{aligned}$$

On the other hand, it follows from (12) that $f''(1/2)=4[7\zeta(3)-\pi^2\log 2]/\pi=2.0032\cdots>0$. Hence f' is not monotone on $(0,1/2]$, so that f is neither convex nor concave on $(0,1/2]$.

b) For each $t\in(0,1)$ and for $x\in(0,1/2]$, let $g_1(x)=t^{x-1}+t^{-x}$. It is easy to show that g_1 is strictly decreasing and convex from $(0,1/2]$ onto $[2/\sqrt{t},(1+t)/t)$, $g_1^{(2n)}$ ($g_1^{(2n-1)}$) is strictly decreasing (increasing) and convex (concave, respectively) on $(0,1/2]$ for $n\in\mathbf{N}$.

By (5), $R(x)$ and $\mathrm{B}(x)$ can be rewritten as

$$R(x)=\int_0^1\{[g_1(x)-2]/(1-t)\}\mathrm{d}t, \tag{17}$$

$$\mathrm{B}(x)=\int_0^1\frac{t^{x-1}}{1+t}\mathrm{d}t+\int_1^{\infty}\frac{t^{x-1}}{1+t}\mathrm{d}t=\int_0^1\frac{g_1(x)}{1+t}\mathrm{d}t, \tag{18}$$

respectively, and hence

$$g(x)=\int_0^1\frac{[1-c+(1+c)t]g_1(x)-2(1+t)}{1-t^2}\mathrm{d}t. \tag{19}$$

If $c\leqslant 1$, then by (19), g is strictly decreasing on $(0,1/2]$. If $c>1$, then

$$g'(0^+)=\lim_{x\to 0}\left[\frac{\pi x}{\sin(\pi x)}\right]^2\frac{c\cos(\pi x)-\{[\sin(\pi x)]/(\pi x)\}^2}{x^2}=+\infty,$$

so that g is not decreasing on $(0,1/2]$, since

$$g'(x) = \psi'(1-x) - \psi'(x+1) - \frac{1}{x^2} + c\pi^2 \frac{\cos(\pi x)}{\sin^2(\pi x)} \tag{20}$$

by (3). Thus, g is strictly decreasing on $(0,1/2]$ if and only if $c \leqslant 1$.

If $c < 1$, then $g(1/2) = \log 16 - c\pi$, and

$$\begin{aligned} g(0^+) &= \lim_{x\to 0}\left\{-[\gamma+\psi(1-x)] - [\gamma+\psi(1+x)] + \frac{1}{x} - \frac{c\pi}{\sin(\pi x)}\right\} \\ &= \lim_{x\to 0}\frac{\sin(\pi x) - c\pi x}{x\sin(\pi x)} = \lim_{x\to 0}\frac{\pi x}{\sin(\pi x)}\frac{[\sin(\pi x)]/(\pi x) - c}{x} = +\infty. \end{aligned}$$

If $c = 1$, then $g(1/2) = \log 16 - \pi = -\beta/2$, and $g(0^+) = 0$ by (19).

If $c \leqslant 1$, then by (19), g' is strictly increasing on $(0,1/2]$ so that g is convex on $(0,1/2]$. On the other hand, by(20),

$$g''(x) = -\psi''(1-x) - \psi''(x+1) + \frac{2}{x^3} - c\pi^3 \frac{1+\cos^2(\pi x)}{\sin^3(\pi x)} \tag{21}$$

so that for $c > 1$,

$$\begin{aligned} g''(0^+) &= -2\psi''(1) + \lim_{x\to 0}\frac{2\sin^3(\pi x) - c(\pi x)^3[1+\cos^2(\pi x)]}{[x\sin(\pi x)]^3} \\ &= 4\zeta(3) + \lim_{x\to 0}\frac{2\{[\sin(\pi x)]/(\pi x)\}^3 - c[1+\cos^2(\pi x)]}{x^3} = -\infty, \end{aligned}$$

and hence g is not convex on $(0,1/2]$ if $c > 1$. Consequently, g is convex $(0,1/2]$ if and only if $c \leqslant 1$.

The assertion for $g^{(n)}$ follows from the following expression

$$g^{(n)}(x) = \int_0^1 \frac{[1-c+(1+c)t]g_1^{(n)}(x)}{1-t^2}\mathrm{d}t.$$

If $1 < c < 28\zeta(3)/\pi^3$, then by (20) and (21),

$$g'(0^+) = +\infty, g'(1/2) = 0, g''(0^+) = -\infty, g''(1/2) = 28\zeta(3) - c\pi^3 > 0.$$

This shows that there exist x_1 and x_2 such that $g'(x) > 0$ for $0 < x < x_1$ and $g'(x) < 0$ for $x_2 < x < 1/2$. Hence for $1 < c < 28\zeta(3)/\pi^3$, g is not monotone on $(0,1/2]$.

c) Differentiation and (19) give

$$h'(x) = \frac{(1-2x)g'(x) + 4g(x)}{(1-2x)^3} = \frac{h_1(x)}{h_2(x)} \tag{22}$$

where $h_1(x) = (1-2x)g'(x) + 4g(x)$ and $h_2(x) = (1-2x)^3$. Clearly, $h_1{}'(x) = 2g'(x) + (1-2x)g''(x)$, $h_1{}''(x) = (1-2x)g'''(x)$, $g(1/2) = 0$ for $c = \alpha$, and $g'(1/2) = 0$ since

$$g'(x) = \int_0^1 \frac{[1-c+(1+c)t]g_1{}'(x)}{1-t^2}\mathrm{d}t.$$

Hence $h_1(1/2) = h_2(1/2) = h_1{}'(1/2) = h_2{}'(1/2) = 0$. Since

$$h_1{}''(x)/h_2{}''(x) = g'''(x)/24 \tag{23}$$

which is strictly increasing in x on $(0,1/2]$ by Theorem 1(b). Therefore, h' is strictly increasing in x on $(0,1/2]$ by (22) − (23) and the Monotone l'Hôpital's Rule [3, Theorem 1.25], so that h is convex $(0,1/2]$. Since

$$\lim_{x\to 1/2} h'(x) = \frac{1}{24}\int_0^1 \frac{1-c+(1+c)t}{1-t^2} g_1{}'''(x)\mathrm{d}t = 0,$$

the monotonicity of h follows. Clearly, $h(0^+) = g(0^+) = +\infty$. By l'Hôpital's Rule,

$$\begin{aligned} h\left(\frac{1}{2}\right) &= \frac{1}{4}\left[\lim_{x\to 1/2}\frac{\psi'(1-x) - \psi'(x)}{2x-1} + \pi(\log 16)\lim_{x\to 1/2}\frac{\cos(\pi x)}{2x-1}\right] \\ &= [-2\psi''(1/2) - \pi^2\log 16]/8 = [7\zeta(3) - \pi^2\log 2]/2. \end{aligned}$$

The inequality (8) and its equality case are clear.

2 Proof of Theorem 2

a) Clearly, $F(x)=-\frac{g(x)}{x}$ with $c=1$, and hence the monotonicity of F follows from [3, Theorem 1.25] and the convexity of g. Now let $F_1(x)=g(x)-xg'(x)$ and $F_2(x)=x^2$. Then $F_1(0^+)=F_2(0)=0$, $F'(x)=F_1(x)/F_2(x)$, and by (19),

$$F_1{}'(x)/F_2{}'(x)=-g''(x)/2=-\int_0^1[t(\log t)^2 g_1(x)/(1-t^2)]\mathrm{d}t \tag{24}$$

which is strictly increasing on $(0,1/2]$. This shows that F' is strictly increasing on $(0,1/2]$ by [3, Theorem 1.25], and hence the convexity of F follows.

Clearly, $F(1/2)=\beta$. By (20) and l'H$\hat{\ }$opital's Rule, we obtain

$$\begin{aligned} F(0^+) &= \lim_{x\to 0}\frac{\sin^2(\pi x)-(\pi x)^2\cos(\pi x)}{[x\sin(\pi x)]^2}=\frac{1}{\pi^2}\lim_{x\to 0}\frac{\sin^2(\pi x)-(\pi x)^2\cos(\pi x)}{x^4} \\ &= \frac{\pi^2}{4}+\frac{1}{2\pi}\lim_{x\to 0}\frac{\sin(\pi x)-\pi x}{x^3}\cos(\pi x)=\frac{\pi^2}{4}+\frac{1}{6}\lim_{x\to 0}\frac{\cos(\pi x)-1}{x^2}=\frac{\pi^2}{6}. \end{aligned}$$

The double inequality (9) and its equality case are clear.

By (19),

$$F(x)=\frac{2}{x}\int_0^1\frac{1}{1-t^2}[1+t-t(t^{x-1}+t^{-x})]\mathrm{d}t \tag{25}$$

$$F'(x)=\frac{2}{x^2}\int_0^1\frac{1}{1-t^2}[(t^x+t^{1-x})-x(t^x-t^{1-x})\log t-(1+t)]\mathrm{d}t \tag{26}$$

$$F''(x)=2\int_0^1[F_3(x)/(1-t^2)]\mathrm{d}t \tag{27}$$

where $F_3(x)=F_4(x)/F_5(x)$, $F_4(x)=2(1+t)+2x(t^x-t^{1-x})\log t-2(t^x+t^{1-x})-x^2(t^x+t^{1-x})(\log t)^2$ and $F_5(x)=x^3$. Clearly, $F_4(0)=F_5(0)=0$. Differentiation gives

$$F_4{}'(x)/F_5{}'(x)=[(t^x-t^{1-x})(\log t)^2\log(1/t)]/3,$$

which is strictly decreasing in x on $(0,1/2]$. By [3, Theorem 1.25], this shows that F_3 is strictly decreasing in x on $(0,1/2]$, and hence the concavity of F' follows from (27).

Clearly, $F'(1/2)=F_1(1/2)/F_2(1/2)=-2\beta$. Since $\sin^3(\pi x)=\pi^3[x^3-(\pi^2/2)x^5+O(x^7)]$ and $\cos^2(\pi x)=1-(\pi x)^2+O(x^4)$ as $x\to 0$, it follows from (24) and (21) that

$$\begin{aligned} F'(0^+) &= -g''(0^+)/2=\psi''(1)-\frac{1}{2\pi^3}\lim_{x\to 0}\frac{2\sin^3(\pi x)-(\pi x)^3[1+\cos^2(\pi x)]}{x^6}\left[\frac{\pi x}{\sin(\pi x)}\right]^3 \\ &= \psi''(1)=-2\zeta(3). \end{aligned}$$

b) Clearly, $G(x)$ can be rewritten as

$$G(x)=F(x)+(1-\alpha)\frac{x\mathrm{B}(x)-1}{x^2}. \tag{28}$$

It is well known that for $|t|<\pi$,

$$\frac{1}{\sin t}=\frac{1}{t}+2\sum_{n=1}^{\infty}(1-2^{1-2n})\pi^{-2n}\zeta(2n)t^{2n-1}. \tag{29}$$

(See [1, 4.3.68 & 23.1.18].) Hence

$$\frac{x\mathrm{B}(x)-1}{x^2}=2\sum_{n=1}^{\infty}(1-2^{1-2n})\zeta(2n)x^{2(n-1)}=2\sum_{n=0}^{\infty}(1-2^{-2n-1})\zeta(2n+2)x^{2n} \tag{30}$$

by (7), and it follows from (28) and (30) that

$$G'(x)=F'(x)+(1-\alpha)\frac{x^2\mathrm{B}'(x)-x\mathrm{B}(x)+2}{x^3}=F'(x)+4(1-\alpha)\sum_{n=1}^{\infty}n(1-2^{-2n-1})\zeta(2n+2)x^{2n-1}. \tag{31}$$

By part(1), F' is strictly increasing on $(0,1/2]$. Therefore, (31) shows that G' is strictly increasing on $(0,1/2]$, and hence the convexity of G follows.

By part (1) and the first equality in (31), we obtain the value $G'(1/2) = -4\beta[1-(2/\pi)] < 0$. Hence $G'(x) < 0$ for all $x \in (0,1/2]$ by the monotonicity of G'. This yields the monotonicity of G.

Clearly, $G(1/2) = A_1$. By part(1), (28) and (30), $G(0^+) = A_2$. The double inequality(10) and its equality case are clear.

Remark It is easy to verify that the lower (upper) bound given in (9) is greater than the lower (upper, respectively) bound given in (10). The lower (upper) bound given in (9) ((10), respectively) improves those of $R(x)$ contained in [11, Theorem 2.3]. Theorem 1(a) improves [12, Theorem 2(1)] and [13, Theorem 2(4)] where there was no conclusion on the convexity and/or concavity for the function $f(x) = R(x)/\mathrm{B}(x) = [R(x)\sin(\pi x)]/\pi$, and (8) improves [12, (7)].

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Ramanujan 常数与 Beta 函数的比较

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摘　要: 给出了由 Ramanujan 常数 $R(a)=-2\gamma-\psi(a)-\psi(1-a)$和 Beta 函数 $\mathrm{B}(a,1-a)=\pi/\sin(\pi a)$定义的一些组合的单调性与凹凸性, 获得了 $R(a)-\mathrm{B}(a,1-a)$的一些渐近精确的上下界, 从而深入揭示了函数 $R(a)$与 $\mathrm{B}(a,1-a)$的大小关系, 并改进了 $R(a)$的一些已知的相关结论。

关键词: Ramanujan 常数; psi 和 beta 函数; 单调性; 凹凸性; 上下界

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