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Solution of an extremal problem on the Hübner function

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Abstract: For $r\in(0,1)$, the function $\mathrm{M}(r)=[2r'^2\mathrm{K}(r)\mathrm{K}'(r)/\pi]+\log r$ is known as the Hübner function, where K and K′ are the complete elliptic integrals of the first kind. In this paper, the authors solve an extremal problem on the function M, and present new sharp lower and upper bounds of $\mathrm{M}(r)$, by which some known bounds of $\mathrm{M}(r)$ and the Hersch-Pfluger distortion function $\varphi_K(r)$ for $K\in(0,\infty)$ are improved.

Key words: the Hübner function; extremal problem; Hersch-Pfluger distortion function; complete elliptic integrals; inequality

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Hübner 函数的一个极值问题的解

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摘　要： 对 $r\in(0,1)$，称 $\mathrm{M}(r)=[2r'^2\mathrm{K}(r)\mathrm{K}'(r)/\pi]+\log r$ 为 Hübner 函数，其中 K 和 K′ 为第一类完全椭圆积分。给出了关于 $\mathrm{M}(r)$ 的一个极值问题的解，获得了 $\mathrm{M}(r)$ 的精确上下界，并运用这些结果改进了 $\mathrm{M}(r)$ 和 Hersch-Pfluger 偏差函数 $\varphi_K(r)$ 的已知界。

关键词： Hübner 函数；极值问题；Hersch-Pfluger 偏差函数；完全椭圆积分；不等式

0　Main Results

Throughout this paper, we let B^2 denote the unit disk $\{z\mid z\in C \text{ with } |z|<1\}$, $r'=\sqrt{1-r^2}$ for each $r\in[0,1]$, $QC_K(B^2)=\{f\mid f$ is a K-quasiconformal mapping of B^2 onto itself with $f(0)=0\}$ for $K\geqslant 1$. For $r\in(0,1)$, let K and K′ be the complete elliptic integrals of the first kind defined by

$$\begin{cases}\mathrm{K}=\mathrm{K}(r)=\int_0^{\pi/2}(1-r^2\sin^2 t)^{-1/2}\,\mathrm{d}t\\ \mathrm{K}'=\mathrm{K}'(r)=\mathrm{K}(r')\\ \mathrm{K}(0)=\pi/2,\mathrm{K}(1)=\infty\end{cases}\tag{1}$$

and for $r\in[0,1]$, let E and E′ be the complete elliptic integrals of the second kind defined as

$$\begin{cases}\mathrm{E}=\mathrm{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2 t}\,\mathrm{d}t\\ \mathrm{E}'=\mathrm{E}'(r)=\mathrm{E}(r')\end{cases}\tag{2}$$

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(cf. [1]). For $r \in (0,1)$, put

$$\mathrm{M}(r) = \frac{2}{\pi} r'^2 \mathrm{K}(r)\mathrm{K}'(r) + \log r$$

and let $\mu(r)$ be the conformal modulus of the Grötzsch ring $B^2 \backslash [0,r]$, which is usually called the Grötzsch ring function. It is well known that $\mu(r)$ has the following expression

$$\mu(r) = \pi \mathrm{K}'(r) / [2\mathrm{K}(r)] \tag{3}$$

(cf. [2-3]). For each $K \geqslant 1$ and for $r \in (0,1)$, define the function φ_K on $[0,1]$ by

$$\begin{cases} \varphi_K(r) = \mu^{-1}(\mu(r)/K), r \in (0,1) \\ \varphi_K(0) = \varphi_K(1) - 1 = 0 \end{cases} \tag{4}$$

which is a strictly increasing homeomorphism and called the Hersch-Pfluger distortion function (cf. [1-2]). The special functions K, E, μ and φ_K are indispensable in quasiconformal theory (cf. [1-4]).

In 1952, Hersch and Pfluger extended the well-known classical Schwarz lemma for analytic functions to K-quasiconformal mappings of B^2 into (or onto) itself with the origin fixed, for $K \geqslant 1$ (cf. [1-2, 5]). They proved that for all $z \in B^2$,

$$|f(z)| \leqslant \varphi_K(|z|) \tag{5}$$

if f is a K-quasiconformal mapping of B^2 into itself, and

$$\varphi_{1/K}(|z|) \leqslant |f(z)| \leqslant \varphi_K(|z|) \tag{6}$$

for $f \in QC_K(B^2)$. Each equality in (5)—(6) can be attained. This result is known as the (implicit) quasiconformal Schwarz lemma.

One of the tasks in studying the properties of quasiconformal mappings is to find the lower and upper bounds of $\varphi_K(r)$. It is well known that $\varphi_K(r)$ not only plays an extremely important role in quasiconformal theory [1-3, 6-7], but also has important applications in some other fields of mathematics such as number theory. In number theory, the solutions to Ramanujan's classical modular equations and singular values of complete elliptic integrals can be expressed by $\varphi_K(r)$ (cf. [1, 8-9]). In 1920's, Ramanujan gave numerous algebraic identities satisfied by $\varphi_K(r)$ in his unpublished notebooks (cf. [10]). Hence the research on the bounds of $\varphi_K(r)$ is quite significant. Many mathematicians have obtained various bounds for this function [1-2, 9, 11-15, 19]. The followings are examples for such kind of known results.

In 1960, Wang [16] proved the inequality

$$\varphi_K(r) \leqslant 4^{1-1/K} r^{1/K} \tag{7}$$

for all $r \in (0,1)$ and $K \geqslant 1$. Later, Hübner [17] proved the following sharp inequality

$$\varphi_K(r) \leqslant r^{1/K} \exp((1-1/K)\mathrm{M}(r)) \tag{8}$$

for all $r \in (0,1)$ and $K \geqslant 1$. The inequality (8) is usually called Hübner's inequality, and the function M is sometimes called the Hübner function or the Hübner upper bound function. However, $\mathrm{M}(r)$ is defined in terms of special functions $\mathrm{K}(r)$ and $\mathrm{K}'(r)$. Therefore, many mathematicians have been committed to acquiring the sharp lower and upper bounds for $\varphi_K(r)$ which are given in terms of elementary functions, by obtaining those for $\mathrm{M}(r)$.

In 1999, Qiu et al [7] proved that if $a(r)$, $b(r)$ and $c(r)$ are real functions on $(0,1)$, then the inequalities

$$\varphi_K(r) < r^{1/K} \exp((1-1/K)a(r)) \tag{9}$$

$$\varphi_{1/K}(r) < r^K \exp((1-K)b(r)) \tag{10}$$

$$\varphi_{1/K}(r) > r^K \exp((1-K)c(r)) \tag{11}$$

hold for all $r \in (0,1)$ and $K > 1$ if and only if $a(r) \geqslant \mathrm{M}(r)$, $b(r) \leqslant \mathrm{M}(r)$ and $c(r) \geqslant \mu(r) + \log r$, respectively. This again also shows that the function $\mathrm{M}(r)$ in (8) is best possible. In [7], the authors also found very good approximations for $\mathrm{M}(r)$, and obtained the following sharp bounds of $\mathrm{M}(r)$

$$\max\left\{\frac{(1-r)\operatorname{arth}\sqrt{r}}{\sqrt{r}}\log 4,\frac{r'^2\operatorname{arth}r}{r}\right\}<\mathrm{M}(r)<\min\left\{\frac{2(1-r)\operatorname{arth}\sqrt{r}}{\sqrt{r}},\frac{r'^2\operatorname{arth}r}{r}\log 4\right\} \tag{12}$$

In 2010, Qiu et al [14] improved (12) by proving the following double inequality

$$[(1-\log 4)r+\log 4]\frac{r'^2\operatorname{arth}r}{r}<\mathrm{M}(r)<r'^{3/2}\log 4 \tag{13}$$

for all $r\in(0,1)$. In 2014, Qiu et al [18] improved the upper bound in (13) and solved the following extremal problem: What is the value of

$$\beta=\inf_{0<r<1}\frac{\log(\log 4)-\log\mathrm{M}(r)}{\log(1/(1-r))}?$$

They proved that for $r\in(0,1)$,

$$(1-r)^{\beta}\log 4<\mathrm{M}(r)<r'^{8/5}\log 4 \tag{14}$$

with the best possible constant $\beta=1$. Based on the known results such as those above-mentioned, the following question is natural:

Question 1. What are the best values of α and δ such that

$$(1-r^{\alpha})\log 4<\mathrm{M}(r)<(1-r^{\delta})\log 4 \tag{15}$$

for all $r\in(0,1)$?Or equivalently, what are the values of

$$\begin{cases}\alpha=\inf\limits_{0<r<1}\dfrac{\log(\log 4)-\log(\log 4-\mathrm{M}(r))}{\log(1/r)}\\ \delta=\sup\limits_{0<r<1}\dfrac{\log(\log 4)-\log(\log 4-\mathrm{M}(r))}{\log(1/r)}\end{cases}? \tag{16}$$

The main purpose of this paper is to give the answer to Question 1, by proving our following main result.

Theorem 1. a) For $r\in(0,1)$, let $f(r)=\mathrm{M}(r)+r^{9/5}\log 4$ and $A(r)=\min\{r'^{8/5},1.10239025-r^{9/5}\}$. Then there exists a unique number $r_0\in(0.83,0.84)$ such that the function f is strictly increasing on $(0,r_0]$, and decreasing on $[r_0,1)$, with $f(0^+)=f(1^-)=\log 4$ and $f(r_0)<1.5283$. In particular, for $r\in(0,1)$,

$$(1-r^{9/5})\log 4<\mathrm{M}(r)<A(r)\log 4 \tag{17}$$

b) Let α be defined by (16). Then

$$\begin{cases}\alpha\approx 1.8101\\ 9/5<\alpha<1.81013\end{cases} \tag{18}$$

In particular, for all $r\in(0,1)$, $K>1$, $z\in B^2$ and $f\in QC_K(B^2)$,

$$\varphi_K(r)<r^{1/K}4^{(1-1/K)A(r)} \tag{19}$$

$$\varphi_{1/K}(r)<r^K 4^{(1-K)(1-r^{9/5})} \tag{20}$$

$$|f(z)|\leqslant 4^{(1-1/K)A(r)}|z|^{1/K} \tag{21}$$

c) There is no finite number $\delta\in(0,\infty)$ such that $\mathrm{M}(r)<(1-r^{\delta})\log 4$ holds for all $r\in(0,1)$. More precisely, the value of δ in (16) is ∞.

1 Preliminaries

In this section, we shall prove two lemmas needed in the proof of our main result. First, let us recall the following well-known formulas [8, Theorem 4.1]

$$\begin{cases}\dfrac{\mathrm{dK}}{\mathrm{d}r}=\dfrac{\mathrm{E}-r'^2\mathrm{K}}{rr'^2}\\ \dfrac{\mathrm{dE}}{\mathrm{d}r}=\dfrac{\mathrm{E}-\mathrm{K}}{r}\\ \dfrac{\mathrm{dM}(r)}{\mathrm{d}r}=-\dfrac{4}{\pi r}\mathrm{K}'(\mathrm{K}-\mathrm{E})\end{cases} \tag{22}$$

which will be frequently applied later.

Lemma 1. a) There exists a number $r_1 \in (0.543, 0.544)$ such that the function $f_1(r) \equiv r^{-2.3}[\mathrm{K}(r) - \mathrm{E}(r)]$ is strictly decreasing on $(0, r_1]$, and increasing on $[r_1, 1)$, and the function $f_2(r) \equiv r^{-9/5}\mathrm{K}'(r)[\mathrm{K}(r) - \mathrm{E}(r)]$ is strictly increasing on $[0.544, 1)$.

b) There exists a number $r_2 \in (0.027, 0.028)$ such that the function $f_3(r) \equiv r^{1/5}\mathrm{K}'(r)$ is strictly increasing on $(0, r_2]$, and decreasing on $[r_2, 1)$.

c) The function $f(r) \equiv \mathrm{M}(r) + r^{9/5}\log 4$ is strictly increasing on $(0, 0.081]$.

Proof. a) For $r \in (0,1)$, let $f_4(r) \equiv f_5(r) - 2.3$, where $f_5(r) = r^2\mathrm{E}/[r'^2(\mathrm{K}' - \mathrm{E})]$. Then by differentiation,

$$r^{3.3} f'_1(r) = (\mathrm{K} - \mathrm{E}) f_4(r) \tag{23}$$

By [8, Lemma 5.4(1)], f_4 is strictly increasing from $(0,1)$ onto $(-0.3, \infty)$. Since $f_4(0.543) = -0.00035\cdots$ and $f_4(0.544) = 0.00115\cdots$, the result for f_1 follows from (23).

By [8, Lemma 5.4(1)], the function $r \mapsto \sqrt{r}\,\mathrm{K}'(r)$ is strictly increasing on $(0,1)$. Hence $f_2(r) = \sqrt{r}\,\mathrm{K}' f_1(r)$ is strictly increasing on $[0.544, 1)$, and the result for f_2 follows.

b) Let $f_6(r) = (1/5) - f_7(r)$, where $f_7(r) = (\mathrm{E}' - r^2\mathrm{K}')/(r'^2\mathrm{K}')$. Then by differentiation,

$$r^{4/5} f'_3(r) = \mathrm{K}' f_6(r) \tag{24}$$

By [8, Lemma 5.2(4)], f_6 is strictly decreasing from $(0,1)$ onto $(-3/10, 1/5)$. Since $f_6(0.027) = 0.0002\cdots$ and $f_6(0.028) = -0.0012\cdots$, the conclusion in part b) follows from (24).

c) Let $a = (9\pi\log 2)/10 = 1.959827481273\cdots$, and $f_8(r) = (\mathrm{K} - \mathrm{E})/r^2$ which is strictly increasing from $(0,1)$ onto $(\pi/4, \infty)$ by [8, Lemma 5.2(3)]. Differentiation gives

$$\pi r^{-4/5} f'(r)/4 = f_9(r) \equiv a - r^{-9/5}\mathrm{K}'(\mathrm{K} - \mathrm{E}) = a - f_3(r) f_8(r) \tag{25}$$

By part b), f_9 is strictly decreasing on $(0, 0.027]$, so that

$$f_9(r) \geqslant f_9(0.027) = 0.052788\cdots \text{ for } r \in (0, 0.027] \tag{26}$$

It is clear that for each number $\beta \in (0,2)$, the function $r \mapsto (\mathrm{K} - \mathrm{E})/r^\beta = r^{2-\beta} \cdot (\mathrm{K} - \mathrm{E})/r^2$ is strictly increasing on $(0,1)$. Hence for $r \in (0.027, 0.028]$,

$$f_9(r) > a - \mathrm{K}'(0.027)\frac{\mathrm{K}(0.028) - \mathrm{E}(0.028)}{0.028^{9/5}} = 0.03882\cdots \tag{27}$$

By part b) and [8, Lemma 5.2(2)], for $r \in [0.028, 0.081]$,

$$f_9(r) > a - f_3(0.028) f_8(0.081) = 0.04863\cdots \tag{28}$$

Hence part c) follows from (25)—(28). □

Lemma 2. a) The function $f_{10}(r) \equiv r^{0.426}\mathrm{K}'(r)$ is strictly increasing on $(0, 0.544]$, and the function $f_{11}(r) \equiv r^{0.3}\mathrm{K}'(r)$ is strictly increasing on $(0, 0.15]$.

b) The function $f_{12}(r) \equiv r^{-2.226}[\mathrm{K}(r) - \mathrm{E}(r)]$ is strictly increasing on $[0.5, 1)$, and the function $f_{13}(r) \equiv r^{-2.1}[\mathrm{K}(r) - \mathrm{E}(r)]$ is strictly decreasing on $(0, 0.15]$.

Proof. a) Let $f_{14}(r) = (\mathrm{E}' - r^2\mathrm{K}')/(r'^2\mathrm{K}')$, $f_{15}(r) = 0.426 - f_{14}(r)$ and $f_{16}(r) = 0.3 - f_{14}(r)$. Then

$$\begin{cases} r^{0.574} f_{10}'(r) = \mathrm{K}' f_{15}(r) \\ r^{0.7} f_{11}'(r) = \mathrm{K}' f_{16}(r) \end{cases} \tag{29}$$

By [8, Lemma 5.2(4)], f_{15} and f_{16} are both strictly decreasing on $(0,1)$. Since $f_{15}(0.544) = 0.00067\cdots$ and $f_{16}(0.15) = 0.00289\cdots$, the results in part a) follow from (29).

b) Let f_5 be as in (23). Then we have

$$\begin{cases} r^{3.226} f_{12}'(r) = (\mathrm{K} - \mathrm{E})[f_5(r) - 2.226] \\ r^{3.1} f'_{13}(r) = (\mathrm{K} - \mathrm{E})[f_5(r) - 2.1] \end{cases} \tag{30}$$

Since $f_5(0.5) = 2.24086\cdots > 2.226$ and $f_5(0.15) = 2.01721\cdots < 2.1$, part b) follows from (30). □

2 Proof of Theorem 1

a) The proof of part a) will be completed by our investigating four cases.

Let f_1 be as in Lemma 1, f_9 as in (25), $a=(9\pi\log 2)/10$, and let $f_{17}(r)=\mathrm{K}'(\mathrm{K}-\mathrm{E})/r^{9/5}$. Then $f_{17}(r)=\sqrt{r}\mathrm{K}'(r)f_1(r)$, and by (25),

$$\pi r^{-4/5}f'(r)/4=f_9(r)=a-f_{17}(r) \tag{31}$$

Case (i). $r\in[0.544,1)$.

By [8, Lemma 5.4(1)] and Lemma 1 a), f_{17} is strictly increasing on $[0.544,1)$, and hence f_9 is strictly decreasing on $[0.544,1)$. Since $f_9(0.83)=0.00836\cdots$ and $f_9(0.84)=-0.01943\cdots$, it follows from (31) that there exists a number $r_0\in(0.83,0.84)$ such that f is strictly increasing on $[0.544,r_0]$ and decreasing on $[r_0,1)$.

Case (ii). $r\in[0.5,0.544]$.

We can rewrite $f_{17}(r)$ as $f_{17}(r)=f_{10}(r)f_{12}(r)$. Hence it follows from Lemma 2 that f_{17} is strictly increasing on $[0.5,0.544]$, and f_9 is strictly decreasing on $[0.5,0.544]$ with $f_9(0.544)=0.31378\cdots$. Hence by (31), f is strictly increasing on $[0.5,0.544]$.

Case (iii). $r\in[0.15,0.5]$.

Let $f_{18}(r)=\sqrt{r}\mathrm{K}'(r)$. Then $f_{17}(r)=f_{18}(r)f_1(r)$, and it follows from [8, Lemma 5.4(1)] and Lemma 1 a) that for $r\in[x,y]\subset[0.15,0.5]$,

$$f_9(r)\geqslant f_{19}(x,y)\equiv a-f_1(x)f_{18}(y) \tag{32}$$

Computation gives: $f_{19}(0.22,0.5)=0.03826\cdots$, $f_{19}(0.15,0.22)=0.04048\cdots$. Hence by (32), $f_9(r)>0$ for $r\in[0.15,0.5]$, so that f is strictly increasing on $[0.15,0.5]$ by(31).

Case (iv). $r\in[0.081,0.15]$.

Let f_{11} and f_{13} be as in Lemma 2. Then $f_{17}(r)=f_{11}(r)f_{13}(r)$, and it follows from Lemma 2 and (31) that for $r\in[x,y]\subset[0.081,0.15]$,

$$f_9(r)>a-f_{11}(0.15)f_{13}(0.081)=0.07104\cdots,$$

and hence by (31), f is strictly increasing on $[0.081,0.15]$.

From the above discussion, we see that f is strictly increasing on $[0.081,r_0]$ and decreasing on $[r_0,1)$. This, together with Lemma 1 c), yields the piecewise monotonicity property of f.

Clearly, $f(0^+)=f(1^-)=\log 4$. It is well known that M is strictly decreasing from $(0,1)$ onto $(0,\log 4)$ (cf. [8, Theorem 5.5]), so that

$$f(r_0)<\mathrm{M}(0.83)+0.84^{9/5}\log 4=1.528237374\cdots<1.528237375.$$

Hence by the piecewise monotonicity property of f and (14), we obtain

$$\begin{aligned}(1-r^{9/5})\log 4<\mathrm{M}(r)&<\min\{r'^{8/5}\log 4,(1-r^{9/5})\log 4+f(r_0)-\log 4\}\\&<\min\{r'^{8/5}\log 4,(1-r^{9/5})\log 4+0.1419430131\}\\&=\min\{r'^{8/5},1-r^{9/5}+0.1419430131/\log 4\}\log 4\\&=\min\{r'^{8/5},(1.102390240\cdots)-r^{9/5}\}\log 4<A(r)\log 4,\end{aligned}$$

yielding (17).

b) The first inequality in (18) follows from (17). Taking $r=0.037$, we obtain

$$\frac{\log(\log 4)-\log(\log 4-\mathrm{M}(r))}{\log(1/r)}=1.81012\cdots<1.81013,$$

and hence the second inequality (18) holds.

The inequalities (19) and (20) follow from (9) and (10), while (21) follows from (5) and (19).

c) Let $F(r)=[\log(\log 4)-\log(\log 4-\mathrm{M}(r))]/\log(1/r)$. Then by l'Hôpital's rule,

$$F(1^-)=-\lim_{r\to1}\frac{\mathrm{M}'(r)}{\log4-\mathrm{M}(r)}=\frac{4}{\pi}\lim_{r\to1}\frac{\mathrm{K}'(\mathrm{K}-\mathrm{E})}{\log4-\mathrm{M}(r)}=\frac{1}{\log2}\lim_{r\to1}(\mathrm{K}-\mathrm{E})=\infty \tag{33}$$

It is clear that for all $r\in(0,1)$, $\mathrm{M}(r)<(1-r^{\delta})\log4 \Leftrightarrow \delta\geqslant\sup_{0<r<1}F(r)$. Hence $\delta=\infty$ by (33), and part c) follows. □

Remark. a) Let $g(r)=r'^{8/5}+r^{9/5}-1.10239025$. Then it is easy to show that g is strictly increasing and then decreasing on $(0,1)$. Since $g(0)=g(1)=-0.10239025$ and $g(0.7)=0.007358\cdots$, $r'^{8/5}$ and $1.10239025-r^{9/5}$ in (17) are not directly comparable on $(0,1)$.

b) The function f_{17} in (31) is not monotone on $(0,1)$. As a matter of fact, f_{17} is strictly increasing on $[0.544, 1)$ as shown in the proof of Theorem 1 a). On the other hand, we have $f'_{17}(0.25)=-0.62894\cdots$.

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