浙江理工大学学报(自然科学版),第 37 卷,第 1 期,2017 年 1 月
Journal of Zhejiang Sci-Tech University (Natural Sciences)
Vol. 37, No. 1, Jan. 2017

DOI:10.3969/j.issn.1673-3851.2017.01.017

Series Expansions and Bounds of the Ramanujan Constant

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Abstract: In this paper, the authors present several kinds of series expansions of the Ramanujan constant $R(a)=-2\gamma-\psi(a)-\psi(1-a)$, and monotonicity and convexity properties of certain combinations defined in terms of $R(a)$ and polynomials, according to different application needs. By these results, several asymptotically sharp upper and lower bounds are obtained for $R(a)$, and some known related results can be easily improved. In addition, several identities satisfied by the Riemann zeta function are provided.

Key words: Ramanujan constant; psi function; series expansion; monotonicity and convexity; inequalities

CLC number: O156.4　　Document code: A　　Article ID: 1673-3851 (2017) 01-0104-06

0 Main Results

For complex variable z with Re $z>0$, the gamma and psi functions are defined as

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\mathrm{d}t,\ \psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}, \tag{1}$$

respectively. (Cf. [1-4] for their basic properties.) Throughout this paper, we let $\gamma=0.577215\cdots$ denote the Euler constant as usual. It is well know that the function

$$R(a)\equiv-2\gamma-\psi(a)-\psi(1-a),\ a\in(0,1), \tag{2}$$

which is sometimes called the Ramanujan constant although it is in fact a function of a, is always with the study of zero-balanced Gaussian hypergeometric function $F(a,1\text{-}a;1;z)$. (Cf. [1-3 & 5-13].) By the symmetry, we can assume that $a\in(0,1/2]$ in (2).

It is well known that the function $R(a)$ is essential not only in the study of the zero-balanced Gaussian hypergeometric functions $F(a,1\text{-}a;1;z)$ and the theory of Ramanujan's modular equations, but also in some other fields of mathematics, and its properties are indispensable for us to show properties of $F(a,1\text{-}a;1;z)$ and the functions appearing in Ramanujan's modular equations. (See [1-3 & 5-13].) Some authors have obtained various analytic properties and functional inequalities for this function. (Cf. [3 & 11-16].) In [16], for example, it was proved that $R(x)$ has the following series expansion

$$R(x)=(1/x)+2\sum_{n=1}^{\infty}\zeta(2n+1)x^{2n}. \tag{3}$$

Here and in the sequel, $\zeta(s)=\sum_{k=1}^{\infty}k^{-s}$ denotes the Riemann zeta function as usual. However, the proof of

Received Date: 2016-04-24　　Published Online: 2016-12-09

Fund item: This research is supported by the National Natural Science Foundation of China (NSFC) (1171307)

Introduction of the first author: QIU Songliang (1957-), male, Fuyang, professor, research interests: complex analysis, quasiconformal theoery, special functions, Ramanujan's moduar equations.

(3) given in [16] is too complicated.

We shall also need the following special sum

$$\lambda(n+1) = \sum_{k=0}^{\infty}(2k+1)^{-n-1} \; for \; n \in \mathbf{N}. \tag{4}$$

By [1, 23.2.20], $\lambda(n)$ and $\zeta(n)$ satisfy the following relation

$$\lambda(n) = (1-2^{-n})\zeta(n) \text{ for } n \in \mathbf{N} \text{ with } n \geqslant 2. \tag{5}$$

Motivated by the importance and wide applications of the Ramanujan constant, the authors intend to continue to study the properties of (x) in this paper. Corresponding to different application needs, we shall present several kinds of series expansions for the function $R(x)$, including a very simple proof of (3), and show the monotonicity and convexity properties of certain combinations defined in terms of $R(x)$ and polynomials. By these results, several new asymptotically sharp lower and upper bounds of $R(x)$ are obtained, and some related known results for $R(x)$ can be easily improved. In addition, several identities satisfied by the Riemann zeta function are derived. We now state our main results.

Theorem 1 For $x \in (0,1/2]$ and $n \in \mathbf{N}$, let $y = x(1-x)$, $b_0 = \log 2 - 1$, $c_0 = -1$, $d_0 = \log 4$, and set $a_n = [1+(-1)^n]\zeta(n+1)$,

$$b_n = \frac{1+(-1)^n}{2}\sum_{k=1}^{\infty}\frac{1}{(2k+1)^{n+1}} = \frac{1}{2}[1+(-1)^n][\lambda(n+1)-1],$$

$$c_n = (-1)^n\sum_{k=1}^{\infty}\frac{2k+1}{(k^2+k)^{n+1}}, d_n = [1+(-1)^n]\lambda(n+1).$$

Then we have the following series expansions for R(x):

a) For $x \in (0,1)$,

$$R(x) = \frac{1}{x} + \sum_{n=1}^{\infty}a_n x^n = \frac{1}{x(1-x)} + \sum_{n=1}^{\infty}a_n(1-x)^n. \tag{6}$$

b) For $x \in (0,1/2]$,

$$R(x) = \frac{1}{y} + 4\sum_{n=0}^{\infty}b_n(1-2x)^n = \frac{1}{y} + 4\sum_{n=0}^{\infty}b_{2n}(1-4y)^n. \tag{7}$$

c) For $x \in (0,1/2]$,

$$R(x) = \frac{1}{y} + \sum_{n=0}^{\infty}c_n y^n. \tag{8}$$

d) For $x \in (0,1/2]$,

$$R(x) = 2\sum_{n=0}^{\infty}d_n(1-2x)^n = 2\sum_{n=0}^{\infty}d_{2n}(1-4y)^n = \log 16 + 4\sum_{n=1}^{\infty}\lambda(2n+1)(1-4y)^n. \tag{9}$$

Our next theorem presents some analytic properties of R(x), including its asymptotically sharp lower and upper bounds.

Theorem 2 Let y and b_0 be as in Theorem 1, and set $a = 2\zeta(5) = 2.07385\cdots$, $b = 8[8\log 2 - \zeta(3) - 4] = 2.74496\cdots$, $c = 2\zeta(3) = 2.40411\cdots$, $\alpha = 3 - \log 16 = 0.22741\cdots$, $\beta = \left[\frac{7\zeta(3)}{2}\right] - 4 = 0.20719\cdots$.

Then we have the following conclusions:

a) The function $f(x) \equiv x^{-4}[R(x) - (1/x) - cx^2]$ is strictly increasing and convex from $(0,1/2]$ onto $(a, b]$. In particular, for $x \in (0,1/2]$,

$$(1/x) + cx^2 + ax^4 < R(x) \leqslant (1/x) + cx^2 + ax^4 + 2(b-a)x^5, \tag{10}$$

with equality if and only if $x = 1/2$.

b) The function $g(x) \equiv (1-2x)^{-2}[R(x) - (1/y) - 4b_0]$ is strictly decreasing and convex from $(0,1/2)$ onto (β, α). In particular, for $x \in (0,1/2]$,

$$0\leqslant R(x)-\frac{1}{x(1-x)}+4(1-\log 2)-\beta(1-2x)^2\leqslant(\alpha-\beta)(1-2x)^3, \tag{11}$$

with equality in each instance if and only if $x=1/2$.

c) The function $h(y)\equiv[(1/y)-1-R(x)]/y$ is strictly decreasing and convex from $(0,1/4]$ onto $[4\alpha,1)$. In particular, for $x\in(0,1/2]$,

$$-(1-4\alpha)x(1-x)[1-4x(1-x)]\leqslant R(x)-\frac{1}{x(1-x)}+1+4\alpha x(1-x)\leqslant 0, \tag{12}$$

with equality in each instance if and only if $x=1/2$.

1 Proof of Theorem 1

a) Let $f_1(x)=-\gamma-\psi(1-x)$. Clearly, $f_1(0)=0$. By [1,6.3.5], $R(x)-(1/x)=f_1(x)-[\psi(x+1)+\gamma]$. It is well known that for $n\in\mathbf{N}$,

$$\psi^{(n)}(x)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(x+k)^{n+1}}. \tag{13}$$

(See [1,6.4.10].) By differentiation and (13), we obtain

$$f_1^{(n)}(x)=(-1)^{n+1}\psi^{(n)}(1-x), f_1^{(n)}(0)=(-1)^{n+1}\psi^{(n)}(1)=n!\ \zeta(n+1),$$

so that

$$f_1(x)=f_1(0)+\sum_{n=1}^{\infty}\frac{f_1^{(n)}(0)}{n!}x^n=\sum_{n=1}^{\infty}\zeta(n+1)x^n. \tag{14}$$

On the other hand, by [1,6.3.5,6.3.14 & 6.3.16],

$$\gamma+\psi(x)+\frac{1}{x}=\gamma+\psi(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\zeta(n+1)x^n=\sum_{n=1}^{\infty}\frac{x}{n(n+x)}. \tag{15}$$

Hence the first equality in (6) follows from (14) and (15).

The second equality in (6) holds since $R(1-x)=R(x)$.

b) Let $f_2(x)=R(x)-(1/y)=R(x)-\{(1/x)+[1/(1-x)]\}$. By(2)and(15),we can write $f_2(x)$ as

$$f_2(x)=-[2\gamma+\psi(x+1)+\psi(2-x)]. \tag{16}$$

with $f_2(1/2)=-2[\gamma+\psi(3/2)]=4(\log 2-1)=4b_0$. Differentiation gives

$$f_2^{(n)}(x)=(-1)^{n+1}\psi^{(n)}(2-x)-\psi^{(n)}(x+1)$$

for $n\in\mathbf{N}$, so that by (13),

$$f_2^{(2n-1)}(1/2)=0=b_{2n-1}, f_2^{(2n)}(1/2)=-2\psi^{(2n)}(3/2)=4^{n+1}(2n)!b_{2n}. \tag{17}$$

Hence $f_2(x)$ has the following power series expansion at $x=1/2$:

$$f_2(x)=4b_0+\sum_{n=1}^{\infty}\frac{f_2^{(n)}(1/2)}{n!}\left(x-\frac{1}{2}\right)^n=4\sum_{n=0}^{\infty}b_{2n}(1-2x)^{2n}=4\sum_{n=0}^{\infty}b_n(1-2x)^n. \tag{18}$$

which yields the first equality in (7). The second equality in (7) holds since $(1-2x)^2=1-4y$.

c) Let $f_3(x)=[(1/y)-R(x)-1]/y$, and let

$$f_4(y)=\sum_{k=1}^{\infty}\frac{2k+1}{k(k+1)(k^2+k+y)}.$$

Clearly, $f_4(0)=-c_1$. By(2) and the first equality in (15),

$$f_3(x)=\frac{[\gamma+\psi(x+1)]+[\gamma+\psi((1-x)+1)]-1}{y}. \tag{19}$$

By the third equality in (15), $f_3(x)$ can be rewritten as

$$f_3(x)=\frac{1}{y}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{x}{k+x}+\frac{1-x}{k+1-x}-\frac{1}{k+1}\right)=f_4(y). \tag{20}$$

since $\sum_{k=1}^{\infty}[k(k+1)]^{-1}=1$. Differentiation gives

$$f_4^{(n)}(y) = (-1)^n n!\sum_{k=1}^{\infty}\frac{2k+1}{k(k+1)(k^2+k+y)^{n+1}}. \tag{21}$$

with $\frac{f_4^{(n)}(0)}{n!} = -c_{n+1}$. Hence $f_4(y)$ has the following power series expansion

$$f_4(y) = f_4(0)+\sum_{n=1}^{\infty}\frac{f_4^{(n)}(0)}{n!}y^n = -\sum_{n=0}^{\infty}c_{n+1}y^n = -\sum_{n=1}^{\infty}c_n y^{n-1}. \tag{22}$$

This, together with(20), yields(8).

d)Let $f_5(x) = 1/[x(1-x)]$. Then

$$f_5(x) = \frac{4}{1-(1-2x)^2} = 4\sum_{n=0}^{\infty}(1-2x)^{2n}. \tag{23}$$

and hence by the first equality in (7),

$$\begin{aligned} R(x) &= f_5(x)+4\sum_{n=0}^{\infty}b_{2n}(1-2x)^{2n} = 4\sum_{n=0}^{\infty}(b_{2n}+1)(1-2x)^{2n} \\ &= \log 16+4\sum_{n=1}^{\infty}\lambda(2n+1)(1-2x)^{2n} = 2\sum_{n=0}^{\infty}d_n(1-2x)^n \\ &= \log 16+4\sum_{n=1}^{\infty}\lambda(2n+1)(1-4y)^n = 2\sum_{n=0}^{\infty}d_{2n}(1-4y)^n. \end{aligned}$$

This yields (9).

2 Proof of Theorem 2

a) For $n\in\mathbf{N}$, let a_n be as in Theorem 1. Then $f(x) = x^{-4}[R(x)-(1/x)-a_1x-a_2x^2-a_3x^3]$, and it follows from (6) that

$$f(x) = \sum_{n=4}^{\infty}a_nx^{n-4} = \sum_{n=0}^{\infty}a_{n+4}x^n = 2\sum_{n=0}^{\infty}\zeta(2n+5)x^{2n}. \tag{24}$$

and hence the monotonicity and convexity properties of f follow.

Clearly, $f(1/2) = b$. By(24), $f(0^+) = a$. The double inequality (10) and its equality case are clear.

b) It follows from (7) that

$$g(x) = 4\sum_{n=1}^{\infty}b_{2n}(1-2x)^{2(n-1)} = 4\sum_{n=0}^{\infty}b_{2n+2}(1-2x)^{2n} \tag{25}$$

which yields the monotonicity and convexity properties of g since all the coefficients $b_{2n+2}>0$.

Clearly, $g(0^+) = -1-4b_0 = \alpha$. By(25), $g((1/2)^-) = 4b_2 = \beta$. The double inequality (11) and its equality case follow from the monotonicity and convexity properties of g.

c) It follows from (15) that

$$\begin{aligned} h(y) &= \frac{1}{y}\left[\frac{1}{x}+\frac{1}{1-x}-1+2\gamma+\psi(x)+\psi(1-x)\right] \\ &= \frac{[\gamma+\psi(1+x)]+[\gamma+\psi(2-x)]-1}{y} \\ &= \frac{1}{y}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x}{n+x}+\frac{1-x}{n+1-x}\right)-\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\right] \\ &= \frac{1}{y}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{n+2y}{n^2+n+y}-\frac{1}{n+1}\right) = \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)(n^2+n+y)}, \end{aligned}$$

from which the monotonicity and convexity properties of h follow.

Clearly, $h(1/4) = 4(3-\log 16) = 4\alpha$ and

$$h(0^+) = \sum_{n=1}^{\infty}\frac{2n+1}{(n^2+n)^2} = \sum_{n=1}^{\infty}\frac{(n+1)^2-n^2}{(n^2+n)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2} = 1.$$

The double inequality (12) and its equality case are clear.

Corollary The Riemann zeta function satisfies the following identities

$$\sum_{n=2}^{\infty}(-1)^n\zeta(n)=1, \tag{26}$$

$$\sum_{n=1}^{\infty}4^{-n}\zeta(2n+1)=\log 4-1, \tag{27}$$

$$\sum_{n=1}^{\infty}[(1-2^{-2n-1})\zeta(2n+1)-1]=(3-\log 16)/4, \tag{28}$$

$$\sum_{n=1}^{\infty}[\zeta(2n+1)-1]=1/4. \tag{29}$$

Proof It follows from (15) that

$$R(x)-\frac{1}{y}=-[\gamma+\psi(x+1)]-[\gamma+\psi(1-x+1)]=\sum_{n=1}^{\infty}(-1)^n\zeta(n+1)[x^n+(1-x)^n]. \tag{30}$$

Letting $x\to 0$ in (30), we obtain (26) by (8).

Taking $x=1/2$ in the first equality in (6), we obtain the identity (27). It follows from the first equality in (7) that

$$R(x)-(1/x)=[1/(1-x)]-4(1-\log 2)+4\sum_{n=1}^{\infty}b_{2n}(1-2x)^{2n}. \tag{31}$$

By the first equality in(6), $\lim_{x\to 0}[R(x)-(1/x)]=0$. Hence by letting $x\to 0$ in (31), we obtain

$$\sum_{n=1}^{\infty}b_{2n}=\sum_{n=1}^{\infty}[(1-2^{-2n-1})\zeta(2n+1)-1]=(3-\log 16)/4,$$

so that (28) holds. (29) follows from (27) and (28) .

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Ramanujan 常数的级数展开与上下界

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摘　要: 根据不同的应用需求, 给出了 Ramanujan 常数 $R(a)=-2\gamma-\psi(a)-\psi(1-a)$ 的几类级数展开式、$R(a)$ 与多项式的一些组合的单调性和凹凸性, 并利用这些结果获得了 $R(a)$ 的渐近精确的上下界。运用这些级数展开式, 关于 $R(a)$ 的一些已知结果很容易得到改进。此外还给出了 Riemann zeta 函数满足的几个恒等式。

关键词: Ramanujan 常数; psi 函数; 级数展开; 单调性与凹凸性; 不等式

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